Tetrad Gravity: II) Dirac's Observables.

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Abstract

After a study of the Hamiltonian group of gauge transformations, whose infinitesimal generators are the 14 first class constraints of a new formulation of canonical tetrad gravity on globally hyperbolic, asymptotically flat at spatial infinity, spacetimes with simultaneity spacelike hypersurfaces Σ_τ diffeomorphic to R^3, the multitemporal equations associated with the constraints generating space rotations and space diffeomorphisms on the cotriads are given. Their solutions give the dependence of the cotriads on Σ_τ and of their momenta on the six parameters associated with such transformations. The choice of 3-coordinates on Σ_τ, namely the gauge fixing to the space diffeomorphisms constraints, is equivalent to the choice of how to parametrize the dependence of the cotriad on the last three degrees of freedom: namely to the choice of a parametrization of the superspace of 3-geometries. The Shanmugadhasan canonical transformation, corresponding to the choice of 3-orthogonal coordinates on Σ_τ and adapted to 13 of the 14 first class constraints, is found, the superhamiltonian constraint is rewritten in this canonical basis and the interpretation of the gauge transformations generated by it is given. Some interpretational problems connected with Dirac's observables are discussed. In particular the gauge interpretation of tetrad gravity based on constraint theory implies that a "Hamiltonian kinematical gravitational field" is an equivalence class of pseudo-Riemannian spacetimes modulo the Hamiltonian group of gauge transformations: it includes a conformal 3-geometry and all the different 4-geometries (standard definition of a kinematical gravitational field, $Riem\, M^4/Diff\, M^4$) connected to it by the gauge transformations generated by the constraints, in particular by the superhamiltonian constraint. A “Hamiltonian Einstein or dynamical gravitational field” is a kinematical one which satisfies the Hamilton-Dirac equations generated by the ADM energy: it coincides with the standard Einstein or dynamical gravitational field, namely a 4-geometry solution of Einstein’s equations, since the Hilbert and ADM actions both generate Einstein’s equations so that the kinematical Hamiltonian gauge transformations are dynamically restricted to the spacetime pseudodiffeomorphisms of the solutions of Einstein’s equations. Also the problem of the physical identification of the points of spacetime by means of Komar-Bergmann individuating fields is discussed and some comments on the theory of measurement are done.

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I. INTRODUCTION

In the paper [1] [quoted as I in the following] a new formulation of tetrad gravity was given and the 14 first class constraints of its Hamiltonian description were found. In that paper there was a long Introduction about the research program whose aim is to find a unified description and a canonical reduction of the four interactions based on Dirac-Bergmann theory of constraints. Since the canonical reduction is based on the Shanmugadhasan canonical transformation, in which the original first class constraints are replaced by a subset of the new momenta (whose conjugate variables are Abelianized gauge variables, in the terminology of gauge theories), its use in generally covariant theories has not yet been studied, being associated with a breaking of manifest general covariance. This second paper will explore this approach, because it is the natural one from the point of view of constraint theory (namely presymplectic geometry), like the search of coordinate systems separating the variables is natural in the theory of partial differential equations. There will be a presentation and a (often naive) solution of an ordered sequence of mathematical and interpretational problems, which have to be understood step by step to arrive at a final picture (in a heuristic way at the first stage, when nothing better can be done) and which will require an exact mathematical treatment in future refinements of the theory.

First of all, after a discussion about the parametrization of lapse and shift functions, following the treatment developed for Yang-Mills theories in Ref. [2], in Section II we shall study the Hamiltonian group of gauge transformations whose infinitesimal generators are the 14 first class constraints of tetrad gravity. We shall concentrate, in particular, on the gauge transformations generated by the action of the rotation and space pseudodiffeomorphism (passive diffeomorphisms) constraints on cotriads and on the associated stability groups connected with Gribov ambiguity and isometries.

Then, in Section III, we will define the multitemporal equations associated with the constraints generating space rotations and space pseudodiffeomorphisms [they form a Lie subalgebra of the algebra of gauge transformations]. Their solution allows to find the dependence of cotriads and of their conjugate momenta on the rotation angles and on the three parameters characterizing space pseudodiffeomorphisms (changes of chart in the coordinate atlas of the simultaneity spacelike hypersurface Σ_τ). As a consequence a generic cotriad, which has nine independent degrees of freedom, becomes a function of three angles, of three pseudodiffeomorphims parameters and of three unspecified functions.

In Section IV it is shown that the problem of the choice of the coordinates on the simultaneity spacelike hypersurface Σ_τ is equivalent to the choice of the form of the functional dependence of the cotriad upon these three unspecified functions. The functional dependence corresponding to 3-orthogonal and to normal coordinates around a point on Σ_τ is explicitly given. Since Σ_τ is assumed diffeomorphic to R^3, the 3-orthogonal and normal (around a point) coordinates are globally defined. Then we find the Shanmugadhasan canonical transformation Abelianizing the constraints generating space rotations and space pseudodiffeomorphisms [other seven first class constraints are Abelian from the beginning] in 3-orthogonal coordinates. This allows to get a parametrization of the superspace of 3-geometries in these coordinates.

In Section V a further canonical transformation on the superspace sector, plus its conjugate momenta, allows to put the 3-metric on Σ_τ in a Misner form: the 3-metric in 3-orthogonal coordinates is parametrized by its conformal factor $\phi(\tau,\vec{\sigma}) = e^{q(\tau,\vec{\sigma})/2} = [det\, {}^3g_{rs}(\tau,\vec{\sigma})]^{1/12}$ plus two other variables $r_{\bar{a}}(\tau,\vec{\sigma})$, $\bar{a}=1,2$, whose conjugate momenta are denoted $\pi_\phi(\tau,\vec{\sigma})$, $\pi_{\bar{a}}(\tau,\vec{\sigma})$. Since now 13 of the 14 first class constraints have been transformed in new momenta, we can write the last superhamiltonian constraint in its final form in the 3-orthogonal gauge. This constraint is no more an algebraic relation among the surviving canonical variables (the three parameters labelling 3-geometries and their conjugate momenta: they are the Dirac observables with respect to the gauge transformations generated by 13 constraints, superhamiltonian one excluded), but an integro-differential one for the conformal factor ϕ of the 3-metric (namely the reduced Lichnerowicz equation), because the momenta conjugate to the cotriads are related to the new momenta conjugate to 3-geometries by an integral relation. The last gauge variable of tetrad gravity is not a configurational quantity, but the momentum $\pi_\phi(\tau,\vec{\sigma}) = 2\phi^{-1}(\tau,\vec{\sigma})\,\rho(\tau,\vec{\sigma})$ conjugate to the conformal factor $\phi(\tau,\vec{\sigma})$. This momentum describes a "nonlocal" information on the extrinsic curvature of the spacelike hypersurfaces Σ_τ and replaces the York internal extrinsic time ${}^3K(\tau,\vec{\sigma})$ [in the 3-orthogonal gauge ${}^3K$ is determined by an integral of $\pi_\phi(\tau,\vec{\sigma})$ over all Σ_τ with a nontrivial kernel]. The interpretation of the gauge transformations generated by the superhamiltonian constraint is given.

Therefore, if we add the natural gauge-fixing $\pi_\phi(\tau,\vec{\sigma}) \approx 0$ [$\rho(\tau,\vec{\sigma}) \approx 0$], instead of the maximal slicing condition ${}^3K(\tau,\vec{\sigma}) \approx 0$ of the Lichnerowicz-York conformal approach, we get an identification of the two pairs of canonical variables $r_{\bar{a}}(\tau,\vec{\sigma})$, $\pi_{\bar{a}}(\tau,\vec{\sigma})$, $\bar{a}=1,2$, describing a "Hamiltonian kinematical gravitational field" (the equivalence class of spacetimes modulo the Hamiltonian group of gauge transformations, but not solution of the Hamilton-Dirac equations): it is an equivalence class of 4-geometries (standard kinematical gravitational fields as elements of $Riem\, M^4/Diff\, M^4$) containing a conformal 3-geometry. The reduced ADM energy is playing the role of the physical Hamiltonian for the evolution in the mathematical time parameter labelling the leaves Σ_τ of the foliation of spacetime associated with the chosen 3+1 splitting.

In Section VI there are some conclusions and a discussion of the interpretational problems deriving from the two conflicting point of views based on gauge invariant deterministic Dirac observables and on generally covariant (but not gauge invariant) observables. It is shown that if we define a "Hamiltonian Einstein or dynamical gravitational field" as a kinematical one satisfying the Hamilton-Dirac (and therefore the Einstein) equations, it coincides with the standard Einstein or dynamical gravitational field, namely a 4-geometry solution of Einstein's equations. It is underlined that this is a consequence of the fact that both the Hilbert and ADM actions (even if they have different Noether symmetries) generate the same Einstein equations, so that on the space of their solutions the Hamiltonian gauge transformations are forced to be restricted to the dynamical symmetries of Einstein's equations, namely the spacetime diffeomorphisms of the solutions. Also a discussion of how to give a physical identification of the points of spacetime by means of the Komar-Bergmann individuating fields is given, and some comments on the theory of measurement are done.

In Appendix A there are some notions on coordinate systems. In Appendix B there are some notions concerning isometries and conformal transformations. In Appendix C there is a review of the Lichnerowicz-York conformal approach. In Appendices D and E there is the expression of certain 3- and 4-tensors in the final 3-orthogonal canonical basis of Section V.

II. GAUGE TRANSFORMATION ALGEBRA AND GROUP AND THE STABILITY SUBGROUPS.

As said in I, we shall consider only globally hyperbolic, asymptotically flat at spatial infinity spacetimes M^4 with simultaneity spacelike hypersurfaces Σ_τ (the Cauchy surfaces) diffeomorphic to R^3. The configuration variables of our approach to tetrad gravity are: i) lapse and shift functions $N(\tau,\vec{\sigma})$, $N_{(a)}(\tau,\vec{\sigma})$ [the usual shift functions are $N^r={}^3e^r_{(a)}N_{(a)}$]; ii) boost parameters $\varphi_{(a)}(\tau,\vec{\sigma})$; iii) cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$ on Σ_τ. Their conjugate momenta are $\tilde{\pi}^N(\tau,\vec{\sigma})$, $\tilde{\pi}^{\vec{N}}_{(a)}(\tau,\vec{\sigma})$, $\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma})$, ${}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})$. The fourteen first class constraints and the Dirac Hamiltonian are [$\epsilon=\pm$ according to the chosen signature convention for M^4: $\epsilon(+---)$]

$$\begin{aligned}
\tilde{\pi}^N(\tau,\vec{\sigma}) &\approx 0,\\
\tilde{\pi}^{\vec{N}}_{(a)}(\tau,\vec{\sigma}) &\approx 0,\\
\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) &\approx 0,\\
{}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) &= \frac{1}{2}\epsilon_{(a)(b)(c)}\,{}^3\tilde{M}_{(b)(c)}(\tau,\vec{\sigma}) = \epsilon_{(a)(b)(c)}\,{}^3e_{(b)r}(\tau,\vec{\sigma})\,{}^3\tilde{\pi}^r_{(c)}(\tau,\vec{\sigma}) \approx 0,\\
{}^3\tilde{\Theta}_r(\tau,\vec{\sigma}) &= {}^3\tilde{\pi}^s_{(a)}(\tau,\vec{\sigma})\partial_r\,{}^3e_{(a)s}(\tau,\vec{\sigma}) - \partial_s[{}^3e_{(a)r}(\tau,\vec{\sigma})\,{}^3\tilde{\pi}^s_{(a)}(\tau,\vec{\sigma})] =\\
&= -{}^3e_{(a)r}(\tau,\vec{\sigma})\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) - {}^3\omega_{r(a)}(\tau,\vec{\sigma})\,{}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) \approx 0,\\
\hat{\mathcal{H}}(\tau,\vec{\sigma}) &= \epsilon[k\,{}^3e\,\epsilon_{(a)(b)(c)}\,{}^3e^r_{(a)}\,{}^3e^s_{(b)}\,{}^3\Omega_{rs(c)} -\\
&- \frac{1}{8k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)r}\,{}^3\tilde{\pi}^r_{(b)}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)}](\tau,\vec{\sigma}) \approx 0,
\end{aligned}$$

$$\begin{aligned}
H_{(D)} &= \int d^3\sigma[N\,\hat{\mathcal{H}} - N_{(a)}\hat{\mathcal{H}}_{(a)} + \lambda_N\tilde{\pi}^N + \lambda^{\vec{N}}_{(a)}\tilde{\pi}^{\vec{N}}_{(a)} + \lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)} + \mu_{(a)}\,{}^3\tilde{M}_{(a)}](\tau,\vec{\sigma}) =\\
&= \int d^3\sigma[N\hat{\mathcal{H}} + N^r\,{}^3\tilde{\Theta}_r + \lambda_N\tilde{\pi}^N + \lambda^{\vec{N}}_{(a)}\tilde{\pi}^{\vec{N}}_{(a)} + \lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)} + \hat{\mu}_{(a)}\,{}^3\tilde{M}_{(a)}](\tau,\vec{\sigma}). \qquad (1)
\end{aligned}$$

Here, $(a)=(1),(2),(3)$ is a flat index, while $\sigma^A=\{\tau;\sigma^r\}$ [$A=(\tau,r)$] are Σ_τ-adapted coordinates for M^4. As shown in Section III of I, in each point of Σ_τ quantities like an internal Euclidean vector $V_{(a)}(\tau,\vec{\sigma})$ transform as Wigner spin 1 3-vectors under Lorentz transformations in TM^4 at that point. The constraints ${}^3\tilde{\Theta}_r(\tau,\vec{\sigma})$ are the generators of the extension of space pseudodiffeomorphisms (passive diffeomorphisms) in $Diff\,\Sigma_\tau$ to cotriads on Σ_τ [they replace the secondary constraints $\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma})=\{\partial_r\,{}^3\tilde{\pi}^r_{(a)}-\epsilon_{(a)(b)(c)}\,{}^3\omega_{r(b)}\,{}^3\tilde{\pi}^r_{(c)}\}(\tau,\vec{\sigma})\approx 0$ (SO(3) Gauss laws), see I], while ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma})$ is the generator of space rotations.

Since one of the most important motivations of our approach to tetrad gravity is to arrive at a unified description of the four interactions [1,3], we need to find a solution to the deparametrization problem of general relativity [4]. This means that in the limit of vanishing Newton constant, $G\rightarrow 0$, tetrad gravity plus any kind of matter should go in the description of the given matter in Minkowski spacetime with a 3+1 decomposition based on its foliation with spacelike hypersurfaces, and, in particular, it should be possible to recover the rest-frame Wigner-covariant instant form description of such a matter [5,3]. This was the main reason for the restriction to the above class of spacetimes.

In the next paper [6] we shall study the asymptotic behaviour for $|\vec{\sigma}|\rightarrow\infty$ of the fields of tetrad gravity so that an asymptotic, at spatial infinity, Poincaré algebra of charges exists without problems of supertranslations [7–11] and the asymptotic part of spacetime agrees as much as possible with Minkowski spacetime [our definitions at this preliminary stage will be coordinate dependent, because the Hamiltonian formulation of general relativity is not yet so developed to be able to translate in it coordinate independent statements about asymptotically flat spacetimes [12–27]].

For the time present, however, we must anticipate some of the results of that paper regarding the allowed class of lapse and shift functions: these functions must be parametrized in a form allowing their identification at spatial infinity, in a class of asymptotically Minkowskian coordinate systems, with the flat lapse and shift functions which can be defined in the description of isolated systems in Minkowski spacetime on spacelike hyperplanes.

In Ref. [5] scalar charged particles and electromagnetic fields in Minkowski spacetime were described in parametrized form on an arbitrary foliation of it (3+1 splitting) with spacelike hypersurfaces still denoted Σ_τ, whose points $z^{(\mu)}(\tau,\vec{\sigma})$ [(μ) are flat Cartesian indices] are extra configuration variables with conjugate momenta $\rho_{(\mu)}(\tau,\vec{\sigma})$: this is possible, because, contrary to curved spacetimes, in Minkowski spacetime the transition coefficients $b^{(\mu)}_A = z^{(\mu)}_A$ from arbitrary to Σ_τ-adapted coordinates are flat tetrads defining a holonomic basis of vector fields. Indeed, in each point of Σ_τ the gradients $z^{(\mu)}_A(\tau,\vec{\sigma}) = \partial z^{(\mu)}(\tau,\vec{\sigma})/\partial\sigma^A$ (in Minkowski spacetime we use the notation $A = (\tau;\check{r})$ to conform with Ref. [5]) form a flat tetrad, i.e. ${}^4\eta^{(\mu)(\nu)} = z^{(\mu)}_A\, {}^4g^{AB} z^{(\nu)}_B$ with ${}^4g^{AB}$ being the inverse of the induced 4-metric ${}^4g_{AB} = z^{(\mu)}_A\, {}^4\eta_{(\mu)(\nu)} z^{(\nu)}_B$ on Σ_τ, with the evolution vector given by $z^{(\mu)}_\tau = N_{[z](flat)} l^{(\mu)} + N^{\check{r}}_{[z](flat)} z^{(\mu)}_{\check{r}}$, where $l^{(\mu)}(\tau,\vec{\sigma})$ is the normal to Σ_τ in $z^{(\mu)}(\tau,\vec{\sigma})$ and

$$
\begin{aligned}
&N_{[z](flat)}(\tau,\vec{\sigma}) = \sqrt{{}^4g_{\tau\tau} - {}^3\gamma^{\check{r}\check{s}}\, {}^4g_{\tau\check{r}}\, {}^4g_{\tau\check{s}}} = \sqrt{{}^4g/{}^3\gamma},\\
&N_{[z](flat)\check{r}}(\tau,\vec{\sigma}) = {}^3g_{\check{r}\check{s}}(\tau,\vec{\sigma}) N^{\check{s}}_{[z](flat)}(\tau,\vec{\sigma}) = {}^4g_{\tau\check{r}},
\end{aligned}
$$

are the flat lapse and shift functions defined through the metric like in general relativity [here ${}^3\gamma^{\check{r}\check{u}}\, {}^4g_{\check{u}\check{s}} = \delta^{\check{r}}_{\check{s}}$ with ${}^3\gamma^{\check{r}\check{s}} = -{}^3g^{\check{r}\check{s}}$ of signature (- - -) to conform with the notations of Ref. [5]]; however, they are not independent variables but functionals of $z^{(\mu)}(\tau,\vec{\sigma})$ in Minkowski spacetime. The independence of the description from the choice of the foliation is manifest due to the presence of four first class constraints whose structure is independent from the system under investigation:

$$
\mathcal{H}_{(\mu)}(\tau,\vec{\sigma}) = \rho_{(\mu)}(\tau,\vec{\sigma}) - l_{(\mu)}(\tau,\vec{\sigma}) T^{\tau\tau}_{system}(\tau,\vec{\sigma}) - z_{\check{r}(\mu)}(\tau,\vec{\sigma}) T^{\tau\check{r}}_{system}(\tau,\vec{\sigma}) \approx 0,
$$

where $T^{\tau\tau}_{system}(\tau,\vec{\sigma})$, $T^{\tau\check{r}}_{system}(\tau,\vec{\sigma})$, are the components of the energy-momentum tensor in the holonomic coordinate system on Σ_τ corresponding to the energy- and momentum-density of the isolated system. These four constraints satisfy an Abelian Poisson algebra being solved in four momenta: $\{\mathcal{H}_{(\mu)}(\tau,\vec{\sigma}), \mathcal{H}_{(\nu)}(\tau,\vec{\sigma}^{'})\} = 0$.

The original Dirac Hamiltonian contains a piece given by $\int d^3\sigma \lambda^{(\mu)}(\tau,\vec{\sigma}) \mathcal{H}_{(\mu)}(\tau,\vec{\sigma})$ with $\lambda^{(\mu)}(\tau,\vec{\sigma})$ arbitrary Dirac multipliers. By using ${}^4\eta^{(\mu)(\nu)} = [l^{(\mu)} l^{(\nu)} - z^{(\mu)}_{\check{r}}\, {}^3g^{\check{r}\check{s}} z^{(\nu)}_{\check{s}}](\tau,\vec{\sigma})$ with ${}^3g^{\check{r}\check{s}}$ [inverse of ${}^3g_{\check{r}\check{s}}$] of signature (+++), we can write

$$
\begin{aligned}
&\lambda_{(\mu)}(\tau,\vec{\sigma}) \mathcal{H}^{(\mu)}(\tau,\vec{\sigma}) = [(\lambda_{(\mu)} l^{(\mu)})(l_{(\nu)} \mathcal{H}^{(\nu)}) - (\lambda_{(\mu)} z^{(\mu)}_{\check{r}})({}^3g^{\check{r}\check{s}} z_{\check{s}(\nu)} \mathcal{H}^{(\nu)})](\tau,\vec{\sigma})\\
&\overset{def}{=} N_{(flat)}(\tau,\vec{\sigma})(l_{(\mu)} \mathcal{H}^{(\mu)})(\tau,\vec{\sigma}) - N_{(flat)\check{r}}(\tau,\vec{\sigma})({}^3g^{\check{r}\check{s}} z_{\check{s}(\nu)} \mathcal{H}^{(\nu)})(\tau,\vec{\sigma})
\end{aligned}
$$

with the (nonholonomic form of the) constraints $(l_{(\mu)}\mathcal{H}^{(\mu)})(\tau,\vec{\sigma})\approx 0$, $({}^3g^{\check{r}\check{s}}z_{\check{s}(\mu)}\mathcal{H}^{(\mu)})(\tau,\vec{\sigma})\approx 0$, satisfying the universal Dirac algebra [see the last three lines of Eqs.(7)]. In this way we have defined new flat lapse and shift functions

$$\begin{aligned} N_{(flat)}(\tau,\vec{\sigma}) &= \lambda_{(\mu)}(\tau,\vec{\sigma})l^{(\mu)}(\tau,\vec{\sigma}),\\ N_{(flat)\check{r}}(\tau,\vec{\sigma}) &= \lambda_{(\mu)}(\tau,\vec{\sigma})z^{(\mu)}_{\check{r}}(\tau,\vec{\sigma}). \end{aligned} \tag{2}$$

which have the same content of the arbitrary Dirac multipliers $\lambda_{(\mu)}(\tau,\vec{\sigma})$, namely they multiply primary first class constraints satisfying the Dirac algebra. In Minkowski spacetime they are quite distinct from the previous lapse and shift functions $N_{[z](flat)}$, $N_{[z](flat)\check{r}}$, defined starting from the metric. In general relativity (where the coordinates $z^\mu(\tau,\vec{\sigma})$ do not exist) the lapse and shift functions defined starting from the 4-metric are also the coefficient (in the canonical part of the Hamiltonian) of secondary first class constraints satisfying the Dirac algebra [as shown in I, this is evident both for ADM canonical metric gravity, see Eqs.(77) and (79) of I, and for canonical tetrad gravity, see Eqs.(59), (60) and (62) of I].

Therefore, it is not clear how to arrive at the soldering of tetrad gravity with matter and of the parametrized Minkowski formulation for the same matter. However, when the parametrized Minkowski formulation is restricted to spacelike hyperplanes, the two definitions of lapse and shift functions coincide [and have the same linear grow in $\vec{\sigma}$ as the asymptotic ones of tetrad gravity , in suitable asymptotic Minkowski coordinates, according to existing literature on asymptotic Poincaré charges at spatial infinity [8,9]] and we get a consistent soldering with canonical tetrad gravity if its 3+1 splittings are restricted to have leaves Σ_τ approaching flat spacelike hyperplanes at spatial infinity in a direction-independent way.

Instead if we want to reduce the description of parametrized Minkowski theories to one restricted to flat hyperplanes in Minkowski spacetime, we have to add the gauge-fixings $z^{(\mu)}(\tau,\vec{\sigma})-x^{(\mu)}_s(\tau)-b^{(\mu)}_{\check{r}}(\tau)\sigma^{\check{r}}\approx 0$. Here $x^{(\mu)}_s(\tau)$ denotes a point on the hyperplane Σ_τ chosen as an origin; the $b^{(\mu)}_{\check{r}}(\tau)$'s form an orthonormal triad at $x^{(\mu)}_s(\tau)$ and the τ-independent normal to the family of spacelike hyperplanes is $l^{(\mu)}=b^{(\mu)}_\tau=\epsilon^{(\mu)}{}_{(\alpha)(\beta)(\gamma)}b^{(\alpha)}_{\check{1}}(\tau)b^{(\beta)}_{\check{2}}(\tau)b^{(\gamma)}_{\check{3}}(\tau)$. Each hyperplane is described by 10 configuration variables, $x^{(\mu)}_s(\tau)$, plus the 6 independent degrees of freedom contained in the triad $b^{(\mu)}_{\check{r}}(\tau)$, and by the 10 conjugate momenta: $p^{(\mu)}_s$ and 6 variables hidden in a spin tensor $S^{(\mu)(\nu)}_s$ [5]. With these 20 canonical variables it is possible to build 10 Poincaré generators $\bar{p}^{(\mu)}_s=p^{(\mu)}_s$, $\bar{J}^{(\mu)(\nu)}_s=x^{(\mu)}_sp^{(\nu)}_s-x^{(\nu)}_sp^{(\mu)}_s+S^{(\mu)(\nu)}_s$.

After the restriction to spacelike hyperplanes the previous piece of the Dirac Hamiltonian is reduced to

$$\tilde{\lambda}^{(\mu)}(\tau)\tilde{\mathcal{H}}_{(\mu)}(\tau)-\frac{1}{2}\tilde{\lambda}^{(\mu)(\nu)}(\tau)\tilde{\mathcal{H}}_{(\mu)(\nu)}(\tau),$$

because the time constancy of the gauge-fixings $z^{(\mu)}(\tau,\vec{\sigma})-x^{(\mu)}_s(\tau)-b^{(\mu)}_{\check{r}}(\tau)\sigma^{\check{r}}\approx 0$ implies $\lambda_{(\mu)}(\tau,\vec{\sigma})=\tilde{\lambda}_{(\mu)}(\tau)+\tilde{\lambda}_{(\mu)(\nu)}(\tau)b^{(\nu)}_{\check{r}}\sigma^{\check{r}}$ with $\tilde{\lambda}^{(\mu)}(\tau)=-\dot{x}^{(\mu)}_s(\tau)$, $\tilde{\lambda}^{(\mu)(\nu)}(\tau)=-\tilde{\lambda}^{(\nu)(\mu)}(\tau)=\frac{1}{2}\sum_{\check{r}}[\dot{b}^{(\mu)}_{\check{r}}b^{(\nu)}_{\check{r}}-b^{(\mu)}_{\check{r}}\dot{b}^{(\nu)}_{\check{r}}](\tau)$ [$\dot{}$ means $d/d\tau$]. Since at this stage we have $z^{(\mu)}_{\check{r}}(\tau,\vec{\sigma})\approx b^{(\mu)}_{\check{r}}(\tau)$, so that $z^{(\mu)}_\tau(\tau,\vec{\sigma})\approx N_{[z](flat)}(\tau,\vec{\sigma})l^{(\mu)}(\tau,\vec{\sigma})+N^{\check{r}}_{[z](flat)}(\tau,\vec{\sigma})b^{(\mu)}_{\check{r}}(\tau,\vec{\sigma})\approx\dot{x}^{(\mu)}_s(\tau)+\dot{b}^{(\mu)}_{\check{r}}(\tau)\sigma^{\check{r}}=-\tilde{\lambda}^{(\mu)}(\tau)-\tilde{\lambda}^{(\mu)(\nu)}(\tau)b_{\check{r}(\nu)}(\tau)\sigma^{\check{r}}$, it is only now that we get the coincidence of the two definitions of flat lapse and shift functions, i.e.

$N_{[z](flat)} \approx N_{(flat)}$, $N_{[z](flat)\check{r}} \approx N_{(flat)\check{r}}$.

The description on arbitrary families of spacelike hyperplanes is independent from the choice of the family, due to the 10 first class constraints

$\tilde{\mathcal{H}}^{(\mu)}(\tau) = \int d^3\sigma \mathcal{H}^{(\mu)}(\tau,\vec{\sigma}) = p_s^{(\mu)} -$
$[total\, momentum\, of\, the\, system\, inside\, the\, hyperplane]^{(\mu)} \approx 0,$

$\tilde{\mathcal{H}}^{(\mu)(\nu)}(\tau) = b^{(\mu)}_{\check{r}}(\tau) \int d^3\sigma\, \sigma^{\check{r}} \mathcal{H}^{(\nu)}(\tau,\vec{\sigma}) - b^{(\nu)}_{\check{r}}(\tau) \int d^3\sigma\, \sigma^{\check{r}} \mathcal{H}^{(\mu)}(\tau,\vec{\sigma})$
$= S_s^{(\mu)(\nu)} - [intrinsic\, angular\, momentum\, of\, the\, system\, inside\, the\, hyperplane]^{(\mu)(\nu)}$
$= S_s^{(\mu)(\nu)} - (b^{(\mu)}_{\check{r}}(\tau) l^{(\nu)} - b^{(\nu)}_{\check{r}}(\tau) l^{(\mu)})[boost\, part\, of\, system's\, angular\, momentum]^{\tau\check{r}}$
$-(b^{(\mu)}_{\check{r}}(\tau) b^{(\nu)}_{\check{s}}(\tau) - b^{(\nu)}_{\check{r}}(\tau) b^{(\mu)}_{\check{s}}(\tau))[spin\, part\, of\, system's\, angular\, momentum]^{\check{r}\check{s}} \approx 0.$

Therefore, on spacelike hyperplanes in Minkowski spacetime we have

$$\begin{aligned}
N_{(flat)}(\tau,\vec{\sigma}) &= \lambda_{(\mu)}(\tau,\vec{\sigma}) l^{(\mu)}(\tau,\vec{\sigma}) \mapsto \\
&\mapsto N_{(flat)}(\tau,\vec{\sigma}) = N_{[z](flat)}(\tau,\vec{\sigma}) = \\
&= -\tilde{\lambda}_{(\mu)}(\tau) l^{(\mu)} - l^{(\mu)} \tilde{\lambda}_{(\mu)(\nu)}(\tau) b^{(\nu)}_{\check{s}}(\tau) \sigma^{\check{s}}, \\
N_{(flat)\,\check{r}}(\tau,\vec{\sigma}) &= \lambda_{(\mu)}(\tau,\vec{\sigma}) z^{(\mu)}_{\check{r}}(\tau,\vec{\sigma}) \mapsto \\
&\mapsto N_{(flat)}(\tau,\vec{\sigma}) = N_{[z](flat)\check{r}}(\tau,\vec{\sigma}) = \\
&= -\tilde{\lambda}_{(\mu)}(\tau) b^{(\mu)}_{\check{r}}(\tau) - b^{(\mu)}_{\check{r}}(\tau) \tilde{\lambda}_{(\mu)(\nu)}(\tau) b^{(\nu)}_{\check{s}}(\tau) \sigma^{\check{s}}.
\end{aligned} \tag{3}$$

This is the main difference from the treatment of parametrized Minkowski theories given in Refs. [4]: there, in the phase action (no configuration action is defined), one uses $N_{[z](flat)}$, $N_{[z](flat)\check{r}}$ in place of $N_{(flat)}$, $N_{(flat)\check{r}}$ also on arbitrary spalike hypersurfaces and not only on spacelike hyperplanes.

In Ref. [28] and in the book in Ref. [29] (see also Ref. [8]), Dirac introduced asymptotic Minkowski rectangular coordinates $z^{(\mu)}_{(\infty)}(\tau,\vec{\sigma}) = x^{(\mu)}_{(\infty)}(\tau) + b^{(\mu)}_{(\infty)\,\check{r}}(\tau)\sigma^{\check{r}}$ in M^4 at spatial infinity [here $\{\sigma^{\check{r}}\}$ are the coordinates in an atlas of Σ_τ, not matching the spatial coordinates $z^{(i)}_{(\infty)}(\tau,\vec{\sigma})$]. For each value of τ, the coordinates $x^{(\mu)}_{(\infty)}(\tau)$ labels a point, near spatial infinity chosen as origin. On it there is a flat tetrad $b^{(\mu)}_{(\infty)\,A}(\tau) = (\, l^{(\mu)}_{(\infty)} = b^{(\mu)}_{(\infty)\,\tau} = \epsilon^{(\mu)}{}_{(\alpha)(\beta)(\gamma)} b^{(\alpha)}_{(\infty)\,\check{1}}(\tau) b^{(\beta)}_{(\infty)\,\check{2}}(\tau) b^{(\gamma)}_{(\infty)\,\check{3}}(\tau);\, b^{(\mu)}_{(\infty)\,\check{r}}(\tau)\,)$, with $l^{(\mu)}_{(\infty)}$ τ-independent, satisfying $b^{(\mu)}_{(\infty)\,A}\, {}^4\eta_{(\mu)(\nu)}\, b^{(\nu)}_{(\infty)\,B} = {}^4\eta_{AB}$ for every τ and assumed to be tangent to the boundary $S^2_{\tau,\infty}$ of Σ_τ.

This suggests that, in a suitable class of coordinate systems asymptotic to Minkowski coordinates (for the sake of simplifying the notation the indices $\check{r}$ are replaced with r) and with the general coordinate transformations suitably restricted at spatial infinity so that it is not possible to go outside this class, the lapse and shift functions of tetrad gravity should be parametrized as

$$\begin{aligned}
N(\tau,\vec{\sigma}) &= N_{(as)}(\tau,\vec{\sigma}) + n(\tau,\vec{\sigma}), \qquad n(\tau,\vec{\sigma}) \rightarrow_{|\vec{\sigma}|\rightarrow\infty} 0, \\
N_{(a)}(\tau,\vec{\sigma}) &= N_{(as)(a)}(\tau,\vec{\sigma}) + n_{(a)}(\tau,\vec{\sigma}) =,
\end{aligned}$$

$$= {}^3e^r_{(a)}(\tau,\vec\sigma)[N_{(as)r}(\tau,\vec\sigma)+n_r(\tau,\vec\sigma)], \qquad n_{(a)}(\tau,\vec\sigma)\ \rightarrow_{|\vec\sigma|\rightarrow\infty}\ 0,$$

$$\begin{aligned}
N_{(as)}(\tau,\vec\sigma) &= -\tilde\lambda_{(\mu)}(\tau)l^{(\mu)}_{(\infty)} - l^{(\mu)}_{(\infty)}\tilde\lambda_{(\mu)(\nu)}(\tau)b^{(\nu)}_{(\infty)s}(\tau)\sigma^s = \\
&= -\tilde\lambda_\tau(\tau) - \frac{1}{2}\tilde\lambda_{\tau s}(\tau)\sigma^s,\\
N_{(as)r}(\tau,\vec\sigma) &= -b^{(\mu)}_{(\infty)r}(\tau)\tilde\lambda_{(\mu)}(\tau) - b^{(\mu)}_{(\infty)r}(\tau)\tilde\lambda_{(\mu)(\nu)}(\tau)b^{(\nu)}_{(\infty)s}(\tau)\sigma^s = \\
&= -\tilde\lambda_r(\tau) - \frac{1}{2}\tilde\lambda_{rs}(\tau)\sigma^s. \qquad (4)
\end{aligned}$$

This very strong assumption (which will be studied in more detail in Ref. [6]) implies that we are restricting the allowed 3+1 splittings of M^4 to those whose leaves Σ_τ tend asymptotically at spatial infinity to Minkowski spacelike hyperplanes in a direction-independent way and that only asymptotic coordinate systems are allowed in which the lapse and shift functions have asymptotic behaviours similar to those of Minkowski spacelike hyperplanes; but this is coherent with Dirac's choice of asymptotic rectangular coordinates [modulo 3-diffeomorphisms not changing the nature of the coordinates] and with the assumptions used to define the asymptotic Poincaré charges. In a future paper [30] it will be shown that in this way we can solve the deparametrization problem of general relativity. It is also needed to eliminate consistently supertranslations and coordinate transformations not becoming the identity at spatial infinity [they are not associated with the gravitational fields of isolated systems [31]]. With these assumptions we have from Eqs.(6) of I:

$$\begin{aligned}
&{}^4g_{\tau\tau}(\tau,\vec\sigma) = \epsilon\{[N_{(as)}+n]^2 - [N_{(as)(a)}+n_{(a)}][N_{(as)(a)}+n_{(a)}]\}(\tau,\vec\sigma) = \epsilon\{[N_{(as)}+n]^2 - [N_{(as)r}+\\
&n_r]{}^3e^r_{(a)}\,{}^3e^s_{(a)}[N_{(as)s}+n_s]\}(\tau,\vec\sigma),\\
&{}^4g_{\tau r}(\tau,\vec\sigma) = -\epsilon[{}^3e_{(a)r}(N_{(as)(a)}+n_{(a)})](\tau,\vec\sigma) = -\epsilon[N_{(as)r}+n_r](\tau,\vec\sigma)
\end{aligned}$$

and the following form of the line element

$$\begin{aligned}
ds^2 &= \epsilon\Big([N_{(as)}+n]^2 - [N_{(as)r}+n_r]{}^3e^r_{(a)}\,{}^3e^s_{(a)}[N_{(as)s}+n_s]\Big)(d\tau)^2 - \\
&- 2\epsilon[N_{(as)r}+n_r]d\tau d\sigma^r - \epsilon\,{}^3e_{(a)r}\,{}^3e_{(a)s}d\sigma^r d\sigma^s = \\
&= \epsilon\Big([N_{(as)}+n]^2(d\tau)^2 - [{}^3e_{(a)r}d\sigma^r + (N_{(as)(a)}+n_{(a)})d\tau][{}^3e_{(a)s}d\sigma^s + (N_{(as)(a)}+n_{(a)})d\tau]\Big). \quad (5)
\end{aligned}$$

By using $\tilde\lambda_A(\tau)=\{\tilde\lambda_\tau(\tau);\tilde\lambda_r(\tau)\}$, $\tilde\lambda_{AB}(\tau)=-\tilde\lambda_{BA}(\tau)$, $n(\tau,\vec\sigma)$, $n_{(a)}(\tau,\vec\sigma)$ as new configuration variables [replacing $N(\tau,\vec\sigma)$ and $N_{(a)}(\tau,\vec\sigma)$] in the Lagrangian of I only produces the replacement of the first class constraints $\tilde\pi^N(\tau,\vec\sigma)\approx 0$, $\tilde\pi^{\vec N}_{(a)}(\tau,\vec\sigma)\approx 0$, with the new first class constraints $\tilde\pi^n(\tau,\vec\sigma)\approx 0$, $\tilde\pi^{\vec n}_{(a)}(\tau,\vec\sigma)\approx 0$, $\tilde\pi^A(\tau)\approx 0$, $\tilde\pi^{AB}(\tau)=-\tilde\pi^{BA}(\tau)\approx 0$, corresponding to the vanishing of the canonical momenta conjugate to the new configuration variables [we assume the Poisson brackets $\{\tilde\lambda_A(\tau),\tilde\pi^B(\tau)\}=\delta^B_A$, $\{\tilde\lambda_{AB}(\tau),\tilde\pi^{CD}(\tau)\}=\delta^C_A\delta^D_B-\delta^D_A\delta^C_B$]. The only change in the Dirac Hamiltonian is

$$\int d^3\sigma[\lambda_N\tilde\pi^N + \lambda^{\vec N}_{(a)}\tilde\pi^{\vec N}_{(a)}](\tau,\vec\sigma)\ \mapsto\ \zeta_A(\tau)\tilde\pi^A(\tau) + \zeta_{AB}(\tau)\tilde\pi^{AB}(\tau) + \int d^3\sigma[\lambda_n\tilde\pi^n + \lambda^{\vec n}_{(a)}\tilde\pi^{\vec n}_{(a)}](\tau,\vec\sigma)$$

(the problem of its differentiability and of the needed surface terms [32] will be discussed in Ref. [6]).

A minimal set of (angle independent to avoid supertranslations [9]; this is also in accord with what is needed to define color charges in Yang-Mills theory [2]) boundary conditions on the canonical variables of tetrad gravity, which will be justified in the next paper [6], is [$r = |\vec{\sigma}|$]

$$
\begin{aligned}
&{}^3e_{(a)r}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \delta_{(a)r} + {}^3w_{(a)r}(\tau,\vec{\sigma}),\\
&{}^3w_{(a)r}(\tau,\vec{\sigma}) = \frac{{}^3w_{(as)(a)r}(\tau)}{r} + O(r^{-1}),\\
&{}^3g_{rs}(\tau,\vec{\sigma}) = [{}^3e_{(a)r}\,{}^3e_{(a)s}](\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \delta_{rs} + {}^3h_{rs}(\tau,\vec{\sigma}),\\
&{}^3h_{rs}(\tau,\vec{\sigma}) = \frac{1}{r}\Big[\delta_{(a)r}\,{}^3w_{(as)(a)s}(\tau) + {}^3w_{(as)(a)r}(\tau)\delta_{(a)s}\Big] + O(r^{-2}),\\
&{}^3g^{rs}(\tau,\vec{\sigma}) = [{}^3e^r_{(a)}\,{}^3e^s_{(a)}](\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \delta^{rs} + {}^3h^{rs}(\tau,\vec{\sigma}),\\
&{}^3h^{rs}(\tau,\vec{\sigma}) = \frac{1}{r}\Big[\delta^r_{(a)}\,{}^3w^s_{(as)(a)}(\tau) + {}^3w^r_{(as)(a)}(\tau)\delta^s_{(a)}\Big] + O(r^{-2}),\\
&\\
&{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} {}^3\tilde{p}^r_{(a)}(\tau,\vec{\sigma}),\\
&{}^3\tilde{p}^r_{(a)}(\tau,\vec{\sigma}) = \frac{{}^3\tilde{p}^r_{(as)(a)}(\tau)}{r^2} + O(r^{-3}),\\
&{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) = \frac{1}{4}\Big[{}^3e^r_{(a)}\,{}^3\tilde{\pi}^s_{(a)} + {}^3e^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)}\Big](\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} {}^3\tilde{k}^{rs}(\tau,\vec{\sigma}),\\
&{}^3\tilde{k}^{rs} = \frac{{}^3\tilde{k}^{rs}_{(as)}(\tau)}{r^2} + O(r^{-3}),\\
&{}^3\tilde{k}^{rs}_{(as)}(\tau) = \frac{1}{4}\Big[\delta^r_{(a)}\,{}^3\tilde{p}^s_{(as)(a)} + \delta^s_{(a)}\,{}^3\tilde{p}^r_{(as)(a)}\Big](\tau),\\
&\\
&n(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}),\\
&n_{(a)}(\tau,\vec{\sigma}) = [{}^3e^r_{(a)}n_r](\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \delta^r_{(a)}n_r(\tau,\vec{\sigma}) + O(r^{-(1+\epsilon)}),\\
&n_r(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}),\\
&\\
&\tilde{\pi}^n(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-3}),\\
&\tilde{\pi}^{\vec{n}}_{(a)}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-3}),
\end{aligned}
\tag{6}
$$

No special requirements are needed at this stage for the asymptotic behaviour of the configuration variables $\varphi_{(a)}(\tau,\vec{\sigma})$.

Let us momentarily forget the asymptotic variables $\tilde{\lambda}_A(\tau)$, $\tilde{\lambda}_{AB}(\tau)$ and their conjugate momenta $\tilde{\pi}^A(\tau) \approx 0$, $\tilde{\pi}^{AB}(\tau) \approx 0$. In the 32-dimensional functional phase space $T^*\mathcal{C}$ spanned by the 16 variables $n(\tau,\vec{\sigma})$, $n_{(a)}(\tau,\vec{\sigma})$, $\varphi_{(a)}(\tau,\vec{\sigma})$, ${}^3e_{(a)r}(\tau,\vec{\sigma})$ of the Lagrangian configuration space $\mathcal{C}$ and by their 16 conjugate momenta, we have 14 first class constraints $\tilde{\pi}^n(\tau,\vec{\sigma}) \approx 0$, $\tilde{\pi}^{\vec{n}}_{(a)}(\tau,\vec{\sigma}) \approx 0$, $\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) \approx 0$, ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) \approx 0$, $\hat{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$ and either ${}^3\tilde{\Theta}_r(\tau,\vec{\sigma}) \approx 0$ or $\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) \approx 0$. Seven pairs of conjugate canonical variables, $\{n(\tau,\vec{\sigma}), \tilde{\pi}^n(\tau,\vec{\sigma}); n_{(a)}(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{n}}_{(a)}(\tau,\vec{\sigma}); \varphi_{(a)}(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma})\}$, are already decoupled from the 18-dimensional subspace spanned by $\{{}^3e_{(a)r}(\tau,\vec{\sigma}); {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})\}$. The variables in $\mathcal{C}_g =$

$\{n(\tau,\vec{\sigma}), n_{(a)}(\tau,\vec{\sigma}), \varphi_{(a)}(\tau,\vec{\sigma})\}$ are gauge variables, but due to the decoupling there is no need to introduce gauge-fixing constraints to eliminate them explicitly, at least at this stage.

Therefore, let us concentrate on the reduced 9-dimensional configuration function space $\mathcal{C}_e = \{{}^3e_{(a)r}(\tau,\vec{\sigma})\}$ $[\mathcal{C} = \mathcal{C}_g + \mathcal{C}_e,\ T^*\mathcal{C} = T^*\mathcal{C}_g + T^*\mathcal{C}_e]$ and on the 18-dimensional function phase space $T^*\mathcal{C}_e = \{{}^3e_{(a)r}(\tau,\vec{\sigma}), {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})\}$, on which we have 7 first class constraints ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) \approx 0$, ${}^3\tilde{\Theta}_r(\tau,\vec{\sigma}) \approx 0$, $\hat{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$, whose Poisson brackets, defining an algebra $\bar{g}$, are given in Eqs.(63) of I

$$
\begin{aligned}
\{{}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}^{'})\} &= \epsilon_{(a)(b)(c)}\, {}^3\tilde{M}_{(c)}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
\{{}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_r(\tau,\vec{\sigma}^{'})\} &= {}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}^{'})\, \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^r},\\
\{{}^3\tilde{\Theta}_r(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_s(\tau,\vec{\sigma}^{'})\} &= \Big[{}^3\tilde{\Theta}_r(\tau,\vec{\sigma}^{'})\frac{\partial}{\partial \sigma^s} + {}^3\tilde{\Theta}_s(\tau,\vec{\sigma})\frac{\partial}{\partial \sigma^r}\Big]\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
\{\hat{\mathcal{H}}(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_r(\tau,\vec{\sigma}^{'})\} &= \hat{\mathcal{H}}(\tau,\vec{\sigma}^{'})\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^r},\\
\{\hat{\mathcal{H}}(\tau,\vec{\sigma}), \hat{\mathcal{H}}(\tau,\vec{\sigma}^{'})\} &= \Big[{}^3e^r_{(a)}(\tau,\vec{\sigma})\, \hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) +\\
&+ {}^3e^r_{(a)}(\tau,\vec{\sigma}^{'})\, \hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}^{'})\Big]\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^r} =\\
&= \Big(\Big[{}^3e^r_{(a)}\, {}^3e^s_{(a)}\, [{}^3\tilde{\Theta}_s + {}^3\omega_{s(b)}\, {}^3\tilde{M}_{(b)}]\Big](\tau,\vec{\sigma}) +\\
&+ \Big[{}^3e^r_{(a)}\, {}^3e^s_{(a)}\, [{}^3\tilde{\Theta}_s + {}^3\omega_{s(b)}\, {}^3\tilde{M}_{(b)}]\Big](\tau,\vec{\sigma}^{'})\Big)\, \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^r}.
\end{aligned}
\tag{7}
$$

Let us call $\bar{\mathcal{G}}$ the (component connected to the identity of the) gauge group obtained from successions of gauge transformations generated by these first class constraints. Since ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma})$ [the generators of the inner gauge SO(3)-rotations] and ${}^3\tilde{\Theta}_r(\tau,\vec{\sigma})$ [the generators of space pseudodiffeomorphisms (passive diffeomorphisms) in $Diff\,\Sigma_\tau$ extended to cotriads] form a Lie subalgebra $\bar{g}_R$ of $\bar{g}$ (the algebra of $\bar{\mathcal{G}}$), let $\bar{\mathcal{G}}_R$ be the gauge group without the superhamiltonian constraint and $\bar{\mathcal{G}}_{ROT}$ its invariant subgroup containing only SO(3) rotations. The addition to $\bar{g}_R$ of the superhamiltonian $\hat{\mathcal{H}}(\tau,\vec{\sigma})$ introduces structure functions [the last of Eqs.(7)] as in the ADM Hamiltonian formulation of metric gravity, so that $\bar{g}$ is not a Lie algebra.

The gauge group $\bar{\mathcal{G}}_R$ may be identified with the automorphism group $Aut\,L\Sigma_\tau$ of the trivial principal SO(3)-bundle $L\Sigma_\tau \approx \Sigma_\tau \times SO(3)$ of orthogonal coframes, whose properties are studied in Ref. [33]. The automorphism group $Aut\,L\Sigma_\tau$ contains the structure group SO(3) of $L\Sigma_\tau$ as a subgroup, and, moreover, $Aut\,L\Sigma_\tau$ is itself a principal bundle with base $Diff\,\Sigma_\tau$ (which acts on the base Σ_τ of $L\Sigma_\tau$) and structure group the group of gauge transformations [$Gau\,L\Sigma_\tau$; see Ref. [2] for a review of the notations] of the principal bundle $L\Sigma_\tau$: therefore, locally $Aut\,L\Sigma_\tau$ has the trivialization $[U \subset Diff\,\Sigma_\tau] \times SO(3)$ and we have

$$
\begin{array}{lll}
Aut\,L\Sigma_\tau & \rightarrow & L\Sigma_\tau \approx \Sigma_\tau \times SO(3)\\
\downarrow & & \downarrow\\
Diff\,\Sigma_\tau & \rightarrow & \Sigma_\tau
\end{array}
\tag{8}
$$

Since the geometric nature of the gauge transformations generated by the superhamiltonian constraint in the fixed time Hamiltonian description is different from time diffeomorphisms, see for instance Ref. [34,35], let us concentrate on the study of the non-Abelian algebra $\bar{g}_R$ and of the associated group of gauge transformations $\bar{\mathcal{G}}_R$. Since $\bar{\mathcal{G}}_R$ contains $Diff\,\Sigma_\tau$ (or better its action on the cotriads), it is not a Hilbert-Lie group, at least in standard sense [36,33] (its differential structure is defined in an inductive way); therefore, the standard technology from the theory of Lie groups used for Yang-Mills theory [see Ref. [2] and the appendix of Ref. [37]] is not directly available. However this technology can be used for the invariant subgroup of gauge SO(3)-rotations. The main problem is that it is not clear how to parametrize the group manifold of $Diff\,\Sigma_\tau$: one only knows that its algebra (the infinitesimal diffeomorphisms) is isomorphic to the tangent bundle $T\Sigma_\tau$ [36].

Moreover, while in a Lie (and also in a Hilbert-Lie) group the basic tool is the group-theoretical exponential map (associated with the one-parameter subgroups), which coincides with the geodesic exponential map when the group manifold of a compact semisimple Lie group is regarded as a symmetric Riemann manifold [38], in $Diff\,\Sigma_\tau$ this map does not produce a diffeomorphism between a neighbourhood of zero in the algebra and a neighbourhood of the identity in $Diff\,\Sigma_\tau$ [36,33]. Therefore, to study the Riemannian 3-manifold Σ_τ we have to use the geodesic exponential map as the main tool [39,40], even if it is not clear its relationship with the differential structure of $Diff\,\Sigma_\tau$. The "geodesic exponential map" at $p \in M^4$ sends each vector ${}^4V_p = {}^4V_p^\mu \partial_\mu \in T_pM^4$ at p to the point of unit parameter distance along the unique geodesic through p with tangent vector ${}^4V_p$ at p; in a small neighbourhood U of p the exponential map has an inverse: $q \in U \subset M^4 \Rightarrow q = Exp\,{}^4V_p$ for some ${}^4V_p \in T_pM^4$. Then, ${}^4V_p^\mu$ are the "normal coordinates" x_2^μ of q and U is a "normal neighbourhood" (see Appendix A for a review of special coordinate systems). Let us remark that in this way one defines an inertial observer in free fall at q in general relativity.

In Yang-Mills theory with trivial principal bundles $P(M,G) = M \times G$ [2], the abstract object behind the configuration space is the connection 1-form ω on $P(M,G) = M \times G$ [G is a compact, semisimple, connected, simply connected Lie group with compact, semisimple real Lie algebra g]; instead Yang-Mills configuration space contains the gauge potentials over the base M, ${}^\sigma A^{(\omega)} = \sigma^*\omega$, i.e. the pull-backs to M of the connection 1-form through global cross sections $\sigma : M \rightarrow P$. The group $\mathcal{G}$ of gauge transformations (its component connected to the identity) acting on the gauge potentials on M is interpreted in a passive sense as a change of global cross section at fixed connection ω, ${}^{\sigma_U}A^{(\omega)} = U^{-1}\,{}^\sigma A^{(\omega)}\,U + U^{-1}dU$ (if $\sigma_U = \sigma \cdot U$ with $U : M \rightarrow G$): this formula describes the gauge orbit associated with the given ω. In this case, the group manifold of $\mathcal{G}$ [which is the space of the cross sections of the principal bundle P(M,G)] may be considered the principal bundle $P(M,G) = M \times G$ itself parametrized with a special connection-dependent family of global cross sections, after having chosen canonical coordinates of first kind on a reference fiber (a copy of the group manifold of G) and having parallel (with respect to the given connection) transported them to the other fibers. In this way we avoid the overparametrization of $\mathcal{G}$ by means of the infinite-dimensional space of all possible local and global cross sections from M to P (this would be the standard description of $\mathcal{G}$). The infinitesimal gauge transformations [the Lie algebra $g_{\mathcal{G}}$ of $\mathcal{G}$: it is a vector bundle whose standard fiber is the Lie algebra g] in phase space are generated by the first class constraints giving the Gauss laws $\Gamma_a \approx 0$. By Legendre pullback to configuration space, we find

${}^{\sigma+\delta\sigma}A^{(\omega)} = {}^\sigma A^{(\omega)} + \delta_o\,{}^\sigma A^{(\omega)} = {}^\sigma A^{(\omega)} + U^{-1}(dU + [{}^\sigma A^{(\omega)}, U]) = {}^\sigma A^{(\omega)} + \hat{D}^{(A)}\alpha = {}^\sigma A^{(\omega)} + \{{}^\sigma A^{(\omega)}, \int \alpha_a \Gamma_a\}$ if $U = I + \alpha$.

In our formulation of tetrad gravity the relevant configuration variables are globally defined cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$ on the hypersurface $\Sigma_\tau \approx R^3$, which is a parallelizable Riemannian 3-manifold $(\Sigma_\tau, {}^3g_{rs} = {}^3e_{(a)r}\, {}^3e_{(a)s})$ assumed asymptotically flat (therefore noncompact) at spatial infinity and geodesically complete [so that, due to the Hopf-Rinow theorem [39], every two points of Σ_τ may be connected by a minimizing geodesic segment and there exists a point $p \in \Sigma_\tau$ from which Σ_τ is geodesically complete, that is the geodesic exponential map is defined on the entire tangent space $T_p\Sigma_\tau$]; with these hypotheses we have $T\Sigma_\tau \approx \Sigma_\tau \times R^3$ and the coframe orthogonal principal affine SO(3)-bundle is also trivial $L\Sigma_\tau \approx \Sigma_\tau \times SO(3)$ [its points are the abstract coframes ${}^3\theta_{(a)}$ $(= {}^3e_{(a)r} d\sigma^r$ in global coordinates)]. In the phase space of tetrad gravity the rotations of the structure group SO(3) are generated by the first class constraints ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) \approx 0$. Therefore, in this case the abstract object behind the configuration space is the so(3)-valued soldering 1-form ${}^3\theta = \hat{R}^{(a)}\, {}^3\theta_{(a)}$ [$\hat{R}^{(a)}$ are the generators of the Lie algebra so(3)]. This shows that to identify the global cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$ we have to choose an atlas of coordinate charts on Σ_τ, so that in each chart ${}^3\theta \mapsto \hat{R}^{(a)}\, {}^3e_{(a)r}(\tau,\vec{\sigma}) d\sigma^r$. Since Σ_τ is assumed diffeomorphic to R^3, global coordinate systems exist.

The general coordinate transformations or space pseudodiffeomorphisms of $Diff\, \Sigma_\tau$ are denoted as $\vec{\sigma} \mapsto \vec{\sigma}^{'}(\vec{\sigma}) = \vec{\xi}(\vec{\sigma}) = \vec{\sigma} + \hat{\vec{\xi}}(\vec{\sigma})$; for infinitesimal pseudodiffeomorphisms, $\hat{\vec{\xi}}(\vec{\sigma}) = \delta\vec{\sigma}(\vec{\sigma})$ is an infinitesimal quantity and the inverse infinitesimal pseudodiffeomorphism is $\vec{\sigma}(\vec{\sigma}^{'}) = \vec{\sigma}^{'} - \delta\vec{\sigma}(\vec{\sigma}^{'}) = \vec{\sigma}^{'} - \hat{\vec{\xi}}(\vec{\sigma}^{'})$. The cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$ and the 3-metric ${}^3g_{rs}(\tau,\vec{\sigma}) = {}^3e_{(a)r}(\tau,\vec{\sigma})\, {}^3e_{(a)s}(\tau,\vec{\sigma})$ transform as [see also Eqs.(30) and (31) of I; $\hat{V}(\vec{\xi}(\vec{\sigma}))$ is the operator whose action on functions is $\hat{V}(\vec{\xi}(\vec{\sigma}))f(\vec{\sigma}) = f(\vec{\xi}(\vec{\sigma}))$]

$$
\begin{aligned}
{}^3e_{(a)r}(\tau,\vec{\sigma}) &\mapsto {}^3e^{'}_{(a)r}(\tau,\vec{\sigma}^{'}(\vec{\sigma})) = \frac{\partial\sigma^s}{\partial\sigma^{'r}}\, {}^3e_{(a)s}(\tau,\vec{\sigma}),\\
&\Rightarrow {}^3e_{(a)r}(\tau,\vec{\sigma}) = \frac{\partial\xi^s(\vec{\sigma})}{\partial\sigma^r}\, {}^3e^{'}_{(a)s}(\tau,\vec{\xi}(\vec{\sigma})) = \frac{\partial\xi^s(\vec{\sigma})}{\partial\sigma^r}\, \hat{V}(\vec{\xi}(\vec{\sigma}))\, {}^3e^{'}_{(a)s}(\tau,\vec{\sigma}),
\end{aligned}
$$

$$
{}^3g_{rs}(\tau,\vec{\sigma}) \mapsto {}^3g^{'}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma})) = \frac{\partial\sigma^u}{\partial\sigma^{'r}}\frac{\partial\sigma^v}{\partial\sigma^{'s}}\, {}^3g_{uv}(\tau,\vec{\sigma}),
$$

$$
\begin{aligned}
\delta\, {}^3e_{(a)r}(\tau,\vec{\sigma}) &= {}^3e^{'}_{(a)r}(\tau,\vec{\sigma}^{'}(\vec{\sigma})) - {}^3e_{(a)r}(\tau,\vec{\sigma}) = \delta_o\, {}^3e_{(a)r}(\tau,\vec{\sigma}) + \hat{\xi}^s(\vec{\sigma})\partial_s\, {}^3e_{(a)r}(\tau,\vec{\sigma}) =\\
&= \frac{\partial\sigma^s}{\partial\sigma^{'r}}\, {}^3e_{(a)s}(\tau,\vec{\sigma}) - {}^3e_{(a)r}(\tau,\vec{\sigma}) = -\partial_r\hat{\xi}^s(\vec{\sigma})\, {}^3e_{(a)s}(\tau,\vec{\sigma}),\\
\delta_o\, {}^3e_{(a)r}(\tau,\vec{\sigma}) &= {}^3e^{'}_{(a)r}(\vec{\sigma}) - {}^3e_{(a)r}(\vec{\sigma}) = -[\partial_r\hat{\xi}^s(\vec{\sigma}) + \delta^s_r\hat{\xi}^u(\vec{\sigma})\partial_u]^3e_{(a)s}(\tau,\vec{\sigma}) =\\
&= [\mathcal{L}_{-\hat{\xi}^s\partial_s}\, {}^3e_{(a)u}(\tau,\vec{\sigma})d\sigma^u]_r = -\{{}^3e_{(a)r}(\tau,\vec{\sigma}), \int d^3\sigma_1\hat{\xi}^s(\vec{\sigma}_1)\, {}^3\tilde{\Theta}_s(\tau,\vec{\sigma}_1)\},
\end{aligned}
$$

$$
\begin{aligned}
\delta\, {}^3g_{rs}(\tau,\vec{\sigma}) &= {}^3g^{'}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma})) - {}^3g_{rs}(\tau,\vec{\sigma}) = \delta_o\, {}^3g_{rs}(\tau,\vec{\sigma}) + \hat{\xi}^u(\vec{\sigma})\partial_u\, {}^3g_{rs}(\tau,\vec{\sigma}) =\\
&= \frac{\partial\sigma^u}{\partial\sigma^{'r}}\frac{\partial\sigma^v}{\partial\sigma^{'s}}\, {}^3g_{uv}(\tau,\vec{\sigma}) - {}^3g_{rs}(\tau,\vec{\sigma}) = -[\delta^u_r\partial_s\hat{\xi}^v(\vec{\sigma}) + \delta^v_s\partial_r\hat{\xi}^u(\vec{\sigma})]^3g_{uv}(\tau,\vec{\sigma}),\\
\delta_o\, {}^3g_{rs}(\tau,\vec{\sigma}) &= {}^3g^{'}_{rs}(\tau,\vec{\sigma}) - {}^3g_{rs}(\tau,\vec{\sigma}) =\\
&= -\Big[\delta^u_r\partial_s\hat{\xi}^v(\vec{\sigma}) + \delta^v_s\partial_r\hat{\xi}^u(\vec{\sigma}) + \delta^u_r\delta^v_s\hat{\xi}^w(\vec{\sigma})\partial_w\Big]^3g_{uv}(\tau,\vec{\sigma}) =\\
&= [\mathcal{L}_{-\hat{\xi}^w\partial_w}\, {}^3g_{uv}(\tau,\vec{\sigma})d\sigma^u \otimes d\sigma^v]_{rs} = -\{{}^3g_{rs}(\tau,\vec{\sigma}), \int d^3\sigma_1\hat{\xi}^s(\vec{\sigma}_1)\, {}^3\tilde{\Theta}_s(\tau,\vec{\sigma}_1)\}.
\end{aligned}
$$

$$(9)$$

Instead the action of finite and infinitesimal gauge rotations of angles $\alpha_{(c)}(\vec{\sigma})$ and $\delta\alpha_{(c)}(\vec{\sigma})$ is respectively

$$\begin{aligned}
&{}^3e_{(a)r}(\tau,\vec{\sigma}) \mapsto {}^3R_{(a)(b)}(\alpha_{(c)}(\vec{\sigma}))\, {}^3e_{(b)r}(\tau,\vec{\sigma}),\\
&\delta_o\, {}^3e_{(a)r}(\tau,\vec{\sigma}) = \{{}^3e_{(a)r}(\tau,\vec{\sigma}), \int d^3\sigma_1 \delta\alpha_{(c)}(\vec{\sigma}_1)\, {}^3\tilde{M}_{(c)}(\tau,\vec{\sigma}_1)\} =\\
&= \epsilon_{(a)(b)(c)}\delta\alpha_{(b)}(\vec{\sigma})\, {}^3e_{(c)r}(\tau,\vec{\sigma}).
\end{aligned} \tag{10}$$

To identify the algebra $\bar{g}_R$ of $\bar{\mathcal{G}}_R$, let us study its symplectic action on $T^*\mathcal{C}_e$, i.e. the infinitesimal canonical transformations generated by the first class constraints ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma})$, ${}^3\tilde{\Theta}_r(\tau,\vec{\sigma})$. Let us define the vector fields

$$\begin{aligned}
X_{(a)}(\tau,\vec{\sigma}) &= -\{., {}^3\tilde{M}_{(a)}(\tau,\vec{\sigma})\},\\
Y_r(\tau,\vec{\sigma}) &= -\{., {}^3\tilde{\Theta}_r(\tau,\vec{\sigma})\}.
\end{aligned} \tag{11}$$

Due to Eqs.(7) they close the algebra

$$\begin{aligned}
&[X_{(a)}(\tau,\vec{\sigma}), X_{(b)}(\tau,\vec{\sigma}^{'})] = \delta^3(\vec{\sigma},\vec{\sigma}^{'})\epsilon_{(a)(b)(c)}X_{(c)}(\tau,\vec{\sigma}),\\
&[X_{(a)}(\tau,\vec{\sigma}), Y_r(\tau,\vec{\sigma}^{'})] = -\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^{'r}}X_{(a)}(\tau,\vec{\sigma}^{'}),\\
&[Y_r(\tau,\vec{\sigma}), Y_s(\tau,\vec{\sigma}^{'})] = -\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^{'s}}Y_r(\tau,\vec{\sigma}^{'}) - \frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^{'r}}Y_s(\tau,\vec{\sigma}).
\end{aligned} \tag{12}$$

These six vector fields describe the symplectic action of rotation and space pseudodiffeomorphism gauge transformations on the subspace of phase space containing cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$ and their conjugate momenta ${}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})$. The non commutativity of rotations and space pseudodiffeomorphisms means that the action of a space pseudodiffeomorphism on a rotated cotriad produces a cotriad which differ by a rotation with modified angles from the action of the space pseudodiffeomorphism on the original cotriad: if $\vec{\sigma} \rightarrow \vec{\sigma}^{'}(\vec{\sigma})$ is a space-diffeomorphism and ${}^3R_{(a)(b)}(\alpha_{(c)}(\vec{\sigma}))$ is a rotation matrix parametrized with angles $\alpha_{(c)}(\vec{\sigma})$, then

$$\begin{aligned}
&{}^3e_{(a)r}(\tau,\vec{\sigma}) \mapsto {}^3e^{'}_{(a)r}(\tau,\vec{\sigma}^{'}(\vec{\sigma})) = \frac{\partial\sigma^s}{\partial\sigma^{'r}}\, {}^3e_{(a)s}(\tau,\vec{\sigma}),\\
&{}^3R_{(a)(b)}(\alpha_{(c)}(\vec{\sigma})) \qquad {}^3e_{(b)r}(\tau,\vec{\sigma}) \mapsto {}^3R_{(a)(b)}(\alpha^{'}_{(c)}(\vec{\sigma}^{'}(\vec{\sigma})))\, {}^3e^{'}_{(b)r}(\tau,\vec{\sigma}^{'}(\vec{\sigma})) =\\
&\qquad = \frac{\partial\sigma^s}{\partial\sigma^{'r}}\Big[{}^3R_{(a)(b)}(\alpha_{(c)}(\vec{\sigma}))\, {}^3e_{(b)s}(\tau,\vec{\sigma})\Big] =\\
&\qquad = {}^3R_{(a)(b)}(\alpha_{(c)}(\vec{\sigma}))\, {}^3e^{'}_{(b)r}(\vec{\sigma}^{'}(\vec{\sigma})),\\
&\Rightarrow \qquad \alpha^{'}_{(c)}(\vec{\sigma}^{'}(\vec{\sigma})) = \alpha_{(c)}(\vec{\sigma}),
\end{aligned} \tag{13}$$

i.e. the rotation matrices, namely the angles $\alpha_{(c)}(\vec{\sigma})$, behave as scalar fields under space pseudodiffeomorphisms. Under infinitesimal rotations ${}^3R_{(a)(b)}(\delta\alpha_{(c)}(\vec{\sigma})) = \delta_{(a)(b)} + \delta\alpha_{(c)}(\vec{\sigma})(\hat{R}^{(c)})_{(a)(b)} = \delta_{(a)(b)} + \epsilon_{(a)(b)(c)}\delta\alpha_{(c)}(\vec{\sigma})$ and space pseudodiffeomorphisms $\vec{\sigma}^{'}(\vec{\sigma}) =$

$\vec{\sigma} + \delta\vec{\sigma}(\vec{\sigma})$ [$\hat{R}^{(c)}$ are the SO(3) generators in the adjoint representation; $\delta\alpha_{(c)}(\vec{\sigma})$, $\delta\vec{\sigma}(\vec{\sigma})$ are infinitesimal variations], we have

$$\begin{aligned}
\int d^3\sigma_1 d^3\sigma_2 \quad & \delta\sigma^s(\vec{\sigma}_2)\delta\alpha_{(c)}(\vec{\sigma}_1)[Y_s(\tau,\vec{\sigma}_2), X_{(c)}(\tau,\vec{\sigma}_1)]^3e_{(a)r}(\tau,\vec{\sigma}) = \\
&= \int d^3\sigma_2 \delta\beta_{(c)}(\vec{\sigma}_2) X_{(c)}(\tau,\vec{\sigma}_2)^3e_{(a)r}(\tau,\vec{\sigma}), \\
\delta\beta_{(c)}(\vec{\sigma}) = \delta\sigma^s(\vec{\sigma}) & \frac{\partial\alpha_{(c)}(\vec{\sigma})}{\partial\sigma^s}, \\
\Rightarrow \quad & \alpha'_{(c)}(\vec{\sigma}) = \alpha_{(c)}(\vec{\sigma} - \delta\vec{\sigma}(\vec{\sigma})) = \alpha_{(c)}(\vec{\sigma}) - \delta\beta_{(c)}(\vec{\sigma}) \Rightarrow \delta_o\alpha_{(c)}(\vec{\sigma}) = -\delta\beta_{(c)}(\vec{\sigma}).
\end{aligned} \tag{14}$$

The group manifold of the group $\bar{\mathcal{G}}_R$ of gauge transformations [isomorphic to $Aut\, L\Sigma_\tau$] is locally parametrized by $\vec{\xi}(\vec{\sigma})$ and by three angles $\alpha_{(c)}(\vec{\sigma})$ [which are also functions of τ], which are scalar fields under pseudodiffeomorphisms, and contains an invariant subgroup $\bar{\mathcal{G}}_{ROT}$ [the group of gauge transformations of $L\Sigma_\tau$; it is a splitting normal Lie subgroup of $Aut\, L\Sigma_\tau$ [33]], whose group manifold (in the passive interpretation) is the space of the cross sections of the trivial principal bundle $\Sigma_\tau \times SO(3) \approx L\Sigma_\tau$ over Σ_τ, like in SO(3) Yang-Mills theory [2], if Σ_τ is "topologically trivial" (its homotopy groups $\pi_k(\Sigma_\tau)$ all vanish); therefore, it may be parametrized as said above. As affine function space of connections on this principal SO(3)-bundle we shall take the space of spin connection 1-forms ${}^3\omega_{(a)}$, whose pullback to Σ_τ by means of cross sections $\sigma : \Sigma_\tau \rightarrow \Sigma_\tau \times SO(3)$ are the (Levi-Civita) spin connections ${}^3\omega_{r(a)}(\tau,\vec{\sigma})d\sigma^r = \sigma^*\, {}^3\omega_{(a)}$ built with the cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$ [they and not the spin connections are the independent variables of tetrad gravity] such that ${}^3g_{rs} = {}^3e_{(a)r}\, {}^3e_{(a)s}$.

Due to our hypotheses on Σ_τ (parallelizable, asymptotically flat, topologically trivial, geodesically complete), the Hopf-Rinow theorem implies the existence of (at least) one point $p \in \Sigma_\tau$ which can be chosen as reference point and can be connected to every other point $q \in \Sigma_\tau$ with a minimizing geodesic segment γ_{pq}; moreover, the theorem says that the geodesic exponential map Exp_p is defined on all $T_p\Sigma_\tau$. If Σ_τ is further restricted to have sectional curvature ${}^3K_p(\Pi) \leq 0$ for each $p \in \Sigma_\tau$ and each tangent plane $\Pi \subset T_p\Sigma_\tau$, the Hadamard theorem [39] says that for each $p \in \Sigma_\tau$ the geodesic exponential map $Exp_p : T_p\Sigma_\tau \rightarrow \Sigma_\tau$ is a diffeomorphism: therefore, there is a unique geodesic joining any pair of points $p, q \in \Sigma_\tau$ and Σ_τ is diffeomorphic to R^3 as we have assumed.

In absence of rotations, the group $\bar{\mathcal{G}}_R$ is reduced to the group $Diff\, \Sigma_\tau$ of space-diffeomorphisms. In the active point of view, diffeomorphisms are smooth mappings (with smooth inverse) $\Sigma_\tau \rightarrow \Sigma_\tau$: under $Diff\, \Sigma_\tau$ a point $p \in \Sigma_\tau$ is sent (in many ways) in every point of Σ_τ. In the passive point of view, the action of the elements of $Diff\, \Sigma_\tau$, called pseudodiffeomorphisms, on a neighbourhood of a point $p \in \Sigma_\tau$ is equivalent to all the possible coordinatizations of the subsets of the neighbourhood of p [i.e. to all possible changes of coordinate charts containing p].

A coordinate system (or chart) (U, σ) in Σ_τ is a homeomorphism (which is also a diffeomorphism) σ of an open set $U \subset \Sigma_\tau$ onto an open set $\sigma(U)$ of R^3: if $\sigma : U \rightarrow \sigma(U)$ and $p \in U$, then $\sigma(p) = (\sigma^r(p))$, where the functions σ^r are called the coordinate functions of σ. An atlas on Σ_τ is a collection of charts in Σ_τ such that: i) each point $p \in \Sigma_\tau$ is contained in the domain of some chart; ii) any two charts overlap smoothly. Let $\mathcal{A} = \{(U_\alpha, \sigma_\alpha)\}$ be the unique "complete" atlas on Σ_τ, i.e. an atlas by definition containing each coordinate system

$(U_\alpha, \sigma_\alpha)$ in Σ_τ that overlaps smoothly with every coordinate system in $\mathcal{A}$.

Given a diffeomorphism $\phi : \Sigma_\tau \rightarrow \Sigma_\tau$ (i.e. a smooth mapping with smooth inverse) and any chart (U, σ) in $\mathcal{A}$, then $(\phi(U), \sigma_\phi = \sigma \circ \phi^{-1})$ is another chart in $\mathcal{A}$ with $\sigma(U) = \sigma_\phi(\phi(U)) \subset R^3$ and, if $p \in U$ and also $p \in \phi(U)$, $\sigma_\phi(p) = (\sigma \circ \phi^{-1})(p) = \xi(p)$ [i.e. $\vec{\sigma} \mapsto \vec{\sigma}_\phi(\vec{\sigma}) = \vec{\xi}(\vec{\sigma})$]. Therefore, each diffeomorphism $\phi : \Sigma_\tau \rightarrow \Sigma_\tau$ may be viewed as a mapping $\phi_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$. If we consider a point $p \in \Sigma_\tau$ and the set $\mathcal{A}_p = \{(U^p_\beta, \sigma^p_\beta)\}$ of all charts in $\mathcal{A}$ containing p, then for each diffeomorphism $\phi : \Sigma_\tau \rightarrow \Sigma_\tau$ we will have $\phi_{\mathcal{A}} : \mathcal{A}_p \rightarrow \mathcal{A}_p$. This suggests that a local parametrization of $Diff\, \Sigma_\tau$ around a point $p \in \Sigma_\tau$ (i.e. local diffeomorphisms defined on the open sets containing p) may be done by choosing an arbitrary chart (U^p_o, σ^p_o) as the local identity of diffeomorphisms $[\vec{\xi}(\vec{\sigma}) = \vec{\sigma}]$ and associating with every nontrivial diffeomorphism $\phi : \Sigma_\tau \rightarrow \Sigma_\tau$ $[\vec{\sigma} \mapsto \vec{\sigma}^{'}(\vec{\sigma}) = \vec{\xi}(\vec{\sigma})]$ the chart $(U^p_\beta = \phi(U^p_o), \sigma^p_\beta = \sigma^p_o \circ \phi^{-1})$. Since $\Sigma_\tau \approx R^3$ admits global charts Ξ, then the group manifold of $Diff\, \Sigma_\tau$ may be tentatively parametrized (in a nonredundant way) with the space of smooth global cross sections (global coordinate systems) in a fibration $\Sigma_\tau \times \Sigma_\tau \rightarrow \Sigma_\tau$ [each global cross section of this fibration is a copy $\Sigma^{(\Xi)}_\tau$ of Σ_τ with the given coordinate system Ξ]: this is analogous to the parametrization of the gauge group of Yang-Mills theory with a family of global cross sections of the trivial principal bundle $P(M, G) = M \times G$ [see next Section for G=SO(3)]. The infinitesimal pseudodiffeomorphisms [the algebra $T\Sigma_\tau$ of $Diff\, \Sigma_\tau$ [36]; its generators in its symplectic action on $T^*\mathcal{C}_e$ are the vector fields $Y_r(\tau, \vec{\sigma})$] would correctly correspond to the cross sections of the fibration $\Sigma_\tau \times T\Sigma_\tau \rightarrow \Sigma_\tau$. With more general Σ_τ the previous description would hold only locally.

By remembering Eq.(8), the following picture emerges:
i) Choose a global coordinate system Ξ on $\Sigma_\tau \approx R^3$ (for instance 3-orthogonal coordinates).
ii) In the description of $Diff\, \Sigma_\tau$ as $\Sigma_\tau \times \Sigma_\tau \rightarrow \Sigma_\tau$ this corresponds to the choice of a global cross section σ_Ξ in $\Sigma_\tau \times \Sigma_\tau$, chosen as conventional origin of the pseudodiffeomorphisms parametrized as $\vec{\sigma} \mapsto \vec{\xi}(\vec{\sigma})$.
iii) This procedure identifies a cross section $\tilde{\sigma}_\Xi$ of the principal bundle $Aut\, L\Sigma_\tau \rightarrow Diff\, \Sigma_\tau$, whose action on $L\Sigma_\tau$ will be the SO(3) gauge rotations in the chosen coordinate system Ξ on Σ_τ.
iv) This will induce a Ξ-dependent trivialization of $L\Sigma_\tau$ to $\Sigma^{(\Xi)}_\tau \times SO(3)$, in which Σ_τ has Ξ as coordinate system and the identity cross section $\sigma^{(\Xi)}_I$ of $\Sigma^{(\Xi)}_\tau \times SO(3)$ corresponds to the origin of rotations in the coordinate system Ξ (remember that the angles are scalar fields under pseudodiffeomorphisms in $Diff\, \Sigma_\tau$).
v) As we will see in the next Section, it is possible to define new vector fields $\tilde{Y}_r(\tau, \vec{\sigma})$ which commute with the rotations ($[X_{(a)}(\tau, \vec{\sigma}), \tilde{Y}_r(\tau, \vec{\sigma}^{'})] = 0$) and still satisfy the last line of Eqs.(12). In this way the algebra $\bar{g}_R$ of the group $\bar{\mathcal{G}}_R$ is replaced (at least locally) by a new algebra $\bar{g}^{'}_R$, which defines a group $\bar{\mathcal{G}}^{'}_R$, which is a (local) trivialization of $Aut\, L\Sigma_\tau$. It is at this level that the rotations in $\bar{\mathcal{G}}_{ROT}$ may be parametrized with a special family of cross sections of the trivial orthogonal coframe bundle $\Sigma^{(\Xi)}_\tau \times SO(3) \approx L\Sigma_\tau$, as for SO(3) Yang-Mills theory, as said in iv).
We do not know whether these steps can be implemented rigorously in a global way for $\Sigma_\tau \approx R^3$; if this is possible, then the quasi-Shanmugadhasan canonical transformation of Section IV can be defined globally for global coordinate systems on Σ_τ.

Both to study the singularity structure of DeWitt superspace [41–43] for the Riemannian 3-manifolds Σ_τ (the space of 3-metrics ${}^3g$ modulo $Diff\, \Sigma_\tau$), for instance the cone over cone singularities of Ref. [44], and the analogous phenomenon (called in this case Gribov ambiguity) for the group $\bar{\mathcal{G}}_{ROT}$ of SO(3) gauge transformations, we have to analyze the stability subgroups of the group $\bar{\mathcal{G}}_R$ of gauge transformations for special cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$, the basic variables in tetrad gravity. In metric gravity, where the metric is the basic variable and pseudodiffeomorphisms are the only gauge transformations, it is known that if the 3-metric ${}^3g$ over a noncompact 3-manifold like Σ_τ satisfies boundary conditions compatible with being a function in a Sobolev space $W^{2,s}$ with $s>3/2$, then there exist special metrics admitting "isometries" [see Appendix B]. The group $Iso(\Sigma_\tau, {}^3g)$ of isometries of a 3-metric of a Riemann 3-manifold $(\Sigma_\tau, {}^3g)$ is the subgroup of $Diff\,\Sigma_\tau$ which leaves the functional form of the 3-metric ${}^3g_{rs}(\tau,\vec{\sigma})$ invariant [its Lie algebra is spanned by the Killing vector fields]: the pseudodiffeomorphism $\vec{\sigma}\mapsto \vec{\sigma}^{'}(\vec{\sigma})=\vec{\xi}(\vec{\sigma})$ in $Diff\,\Sigma_\tau$ is an isometry in $Iso(\Sigma_\tau, {}^3g)$ [see Eqs.(30) of I] if

$$ {}^3g_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))={}^3g^{'}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))=\frac{\partial\sigma^u}{\partial\sigma^{'r}}\frac{\partial\sigma^v}{\partial\sigma^{'s}}\,{}^3g_{uv}(\tau,\vec{\sigma}). \tag{15} $$

The function space of 3-metrics turns out to be a stratified manifold [42] with singularities. Each stratum contains all metrics ${}^3g$ with the same subgroup $Iso(\Sigma_\tau, {}^3g)\subset Diff\,\Sigma_\tau$ [isomorphic but not equivalent subgroups of $Diff\,\Sigma_\tau$ produce different strata of 3-metrics]; each point in a stratum with n Killing vectors is the vertex of a cone, which is a stratum with n-1 Killing vectors (the cone over cone structure of singularities [44]).

From

$$ {}^3g_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))={}^3g^{'}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))={}^3e^{'}_{(a)r}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))\,{}^3e^{'}_{(a)s}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))=\frac{\partial\sigma^u}{\partial\sigma^{'r}}\frac{\partial\sigma^v}{\partial\sigma^{'s}}\,{}^3g_{uv}(\tau,\vec{\sigma})=\frac{\partial\sigma^u}{\partial\sigma^{'r}}\frac{\partial\sigma^v}{\partial\sigma^{'s}}\,{}^3e_{(a)r}(\tau,\vec{\sigma})\,{}^3e_{(a)s}(\tau,\vec{\sigma}), $$

$$ {}^3e^{'}_{(a)r}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))=R_{(a)(b)}(\gamma(\tau,\vec{\sigma}^{'}(\vec{\sigma})))\frac{\partial\sigma^u}{\partial\sigma^{'r}}\,{}^3e_{(b)u}(\tau,\vec{\sigma}) $$

[at the level of cotriads a pseudodiffeomorphism-dependent rotation is allowed], it follows that also the functional form of the associated cotriads is invariant under $Iso(\Sigma_\tau, {}^3g)$

$$ {}^3e_{(a)r}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))={}^3e^{'}_{(a)r}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))=R_{(a)(b)}(\gamma(\tau,\vec{\sigma}^{'}(\vec{\sigma})))\frac{\partial\sigma^s}{\partial\sigma^{'r}}\,{}^3e_{(b)s}(\tau,\vec{\sigma}). \tag{16} $$

Moreover, ${}^3g_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))={}^3g^{'}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))$ implies ${}^3\Gamma^{'u}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))={}^3\Gamma^{u}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))$ and ${}^3R^{'u}{}_{rst}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))={}^3R^{u}{}_{rst}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))$, so that $Iso(\Sigma_\tau, {}^3g)$ is also the stability group for the associated Christoffel symbols and Riemann tensor

$$ \begin{aligned} {}^3\Gamma^{u}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma})) &= {}^3\Gamma^{'u}_{rs}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))=\\ &= \frac{\partial\sigma^{'u}}{\partial\sigma^v}\frac{\partial\sigma^m}{\partial\sigma^{'r}}\frac{\partial\sigma^n}{\partial\sigma^{'s}}\,{}^3\Gamma^{v}_{mn}(\tau,\vec{\sigma})+\frac{\partial^2\sigma^v}{\partial\sigma^{'r}\partial\sigma^{'s}}\frac{\partial\sigma^{'u}}{\partial\sigma^v},\\ {}^3R^{u}{}_{rst}(\tau,\vec{\sigma}^{'}(\vec{\sigma})) &= {}^3R^{'u}{}_{rst}(\tau,\vec{\sigma}^{'}(\vec{\sigma}))=\\ &= \frac{\partial\sigma^{'u}}{\partial\sigma^v}\frac{\partial\sigma^l}{\partial\sigma^{'r}}\frac{\partial\sigma^m}{\partial\sigma^{'s}}\frac{\partial\sigma^n}{\partial\sigma^{'t}}\,{}^3R^{v}{}_{lmn}(\tau,\vec{\sigma}). \end{aligned} \tag{17} $$

Let us remark that in the Yang-Mills case (see Ref. [2] and the end of this Section) the field strengths have generically a larger stability group (the gauge copies problem) than the gauge potentials (the gauge symmetry problem). Here, one expects that Riemann tensors (the field strengths) should have a stability group $\mathcal{S}_R(\Sigma_\tau, {}^3g)$ generically larger of the one of the Christoffel symbols (the connection) $\mathcal{S}_\Gamma(\Sigma_\tau, {}^3g)$, which in turn should be larger of the isometry group of the metric: $\mathcal{S}_R(\Sigma_\tau, {}^3g) \supseteq \mathcal{S}_\Gamma(\Sigma_\tau, {}^3g) \supseteq Iso(\Sigma_\tau, {}^3g)$. However, these stability groups do not seem to have been explored in the literature.

Since the most general transformation in $\bar{\mathcal{G}}_R$ for cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$, spin connections $\hat{R}^{(a)}\,{}^3\omega_{r(a)}(\tau,\vec{\sigma})$ and field strengths $\hat{R}^{(a)}\,{}^3\Omega_{rs(a)}(\tau,\vec{\sigma})$ is [we send $\Lambda \rightarrow \Lambda^{-1}$ in Eqs.(32) of I to conform with the notations of Ref. [2]]

$$
\begin{aligned}
{}^3e^{\prime R}_{(a)r}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma})) &= {}^3R_{(a)(b)}(\alpha_{(c)}(\tau,\vec{\sigma}))\frac{\partial\sigma^s}{\partial\sigma^{\prime r}}\,{}^3e_{(b)s}(\tau,\vec{\sigma}),\\
\hat{R}^{(a)}\,{}^3\omega^{\prime R}_{r(a)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma})) &= \frac{\partial\sigma^u}{\partial\sigma^{\prime r}}\Big[{}^3R^{-1}(\alpha_{(e)}(\tau,\vec{\sigma}))\,\hat{R}^{(a)}\,{}^3\omega_{u(a)}(\tau,\vec{\sigma})\,{}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))+\\
&+{}^3R^{-1}(\alpha_{(e)}(\tau,\vec{\sigma}))\partial_u\,{}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\Big]=\\
&=\frac{\partial\sigma^u}{\partial\sigma^{\prime r}}\Big[\hat{R}^{(a)}\,{}^3\omega_{u(a)}(\tau,\vec{\sigma})+{}^3R^{-1}(\alpha_{(e)}(\tau,\vec{\sigma}))\,\hat{D}^{(\omega)}_u\,{}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\Big]=\\
&=\hat{R}^{(a)}\,{}^3\omega^{\prime}_{r(a)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma}))+{}^3R^{-1}(\alpha_{(e)}(\tau,\vec{\sigma}))\,\hat{D}^{(\omega^{\prime})}_r\,{}^3R(\alpha^{\prime}_{(e)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma}))),\\
\hat{R}^{(a)}\,{}^3\Omega^{\prime R}_{rs(a)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma})) &= \frac{\partial\sigma^u}{\partial\sigma^{\prime r}}\frac{\partial\sigma^v}{\partial\sigma^{\prime s}}\,{}^3R^{-1}(\alpha_{(e)}(\tau,\vec{\sigma}))\,\hat{R}^{(a)}\,{}^3\Omega_{uv(a)}(\tau,\vec{\sigma})\,{}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))=\\
&=\frac{\partial\sigma^u}{\partial\sigma^{\prime r}}\frac{\partial\sigma^v}{\partial\sigma^{\prime s}}\Big(\hat{R}^{(a)}\,{}^3\Omega_{uv(a)}(\tau,\vec{\sigma})+\\
&+{}^3R^{-1}(\alpha_{(e)}(\tau,\vec{\sigma}))\Big[\hat{R}^{(a)}\,{}^3\Omega_{uv(a)}(\tau,\vec{\sigma}),{}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\Big]\Big)=\\
&=\hat{R}^{(a)}\,{}^3\Omega^{\prime}_{rs(a)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma}))+\\
&+{}^3R^{-1}(\alpha^{\prime}_{(e)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma})))\Big[\hat{R}^{(a)}\,{}^3\Omega^{\prime}_{rs(a)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma})),{}^3R(\alpha^{\prime}_{(e)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma})))\Big].
\end{aligned}
\tag{18}
$$

where $(\hat{D}^{(\omega)}_r)_{(a)(b)} = \hat{D}^{(\omega)}_{(a)(b)r}(\tau,\vec{\sigma}) = \delta_{(a)(b)}\partial_r + \epsilon_{(a)(c)(b)}\,{}^3\omega_{r(c)}(\tau,\vec{\sigma})$ and ${}^3R(\alpha_{(e)})$ are 3×3 rotation matrices, the behaviour of spin connections and field strengths under isometries can be studied.

Let us now briefly review the Gribov ambiguity for the spin connections and the field strengths following Ref. [2]. All spin connections are invariant under gauge transformations belonging to the center Z_3 of SO(3): ${}^3R \in Z_3 \Rightarrow\ {}^3\omega^R_{r(a)} = {}^3\omega_{r(a)}$.

As shown in Ref. [2], there can be special spin connections ${}^3\omega_{r(a)}(\tau,\vec{\sigma})$, which admit a stability subgroup $\bar{\mathcal{G}}^\omega_{ROT}$ ("gauge symmetries") of $\bar{\mathcal{G}}_{ROT}$, leaving them fixed

$$
{}^3R(\alpha_{(e)}(\tau,\vec{\sigma})) \in \bar{\mathcal{G}}^\omega_R \Rightarrow \hat{D}^{(\omega)}_r\,{}^3R(\alpha_{(e)}(\tau,\vec{\sigma})) = 0\ \Rightarrow\ {}^3\omega^R_{r(a)}(\tau,\vec{\sigma}) = {}^3\omega_{r(a)}(\tau,\vec{\sigma}). \tag{19}
$$

From Eq.(16), it follows that under an isometry in $Iso\,(\Sigma_\tau, {}^3g)$ we have ${}^3\omega^{\prime}_{r(a)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma})) = {}^3\omega_{r(a)}(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma}))$, namely the rotations ${}^3R(\gamma(\tau,\vec{\sigma}^{\,\prime}(\vec{\sigma})))$ are gauge symmetries.

When there are gauge symmetries, the spin connection is "reducible": its holonomy group Φ^ω is a subgroup of SO(3) $[\Phi^\omega \subset SO(3)]$ and $\bar{\mathcal{G}}^\omega_R$ [which is always equal to the centralizer of the holonomy group in SO(3), $Z_{SO(3)}(\Phi^\omega)$] satisfies $\bar{\mathcal{G}}^\omega_{ROT} = Z_{SO(3)}(\Phi^\omega) \supset Z_3$.

Moreover, there can be special field strengths ${}^3\Omega_{rs(a)}$ which admit a stability subgroup $\bar{\mathcal{G}}^\Omega_{ROT}$ of $\bar{\mathcal{G}}_{ROT}$ leaving them fixed

$$
\begin{aligned}
{}^3R(\alpha_{(e)}(\tau,\vec{\sigma})\in \bar{\mathcal{G}}^{\Omega}_{R} \;\Rightarrow\; & [\hat{R}^{(a)}\, {}^3\Omega_{rs(a)}(\tau,\vec{\sigma}), {}^3R(\alpha_{(e)}(\tau,\vec{\sigma})] = 0\\
&\Rightarrow {}^3\Omega^{R}_{rs(a)}(\tau,\vec{\sigma}) = {}^3\Omega_{rs(a)}(\tau,\vec{\sigma}).
\end{aligned}
\tag{20}
$$

We have $\bar{\mathcal{G}}^{\Omega}_{ROT} \supseteq \bar{\mathcal{G}}^{\omega}_{ROT} = Z_{SO(3)}(\Phi^{\omega}) \supset Z_3$ and there is the problem of "gauge copies": there exist different spin connections ${}^3\omega_{r(a)}(\tau,\vec{\sigma})$ giving rise to the same field strength ${}^3\Omega_{rs(a)}(\tau,\vec{\sigma})$.

A spin connection is "irreducible", when its holonomy group Φ^{ω} is a "not closed" irreducible matrix subgroup of SO(3). In this case we have $\bar{\mathcal{G}}^{\Omega}_{ROT} \supset \bar{\mathcal{G}}^{\omega}_{ROT} = Z_{SO(3)}(\Phi^{\omega}) = Z_3$ and there are gauge copies, but not gauge symmetries.

Finally, a spin connection ${}^3\omega_{r(a)}(\tau,\vec{\sigma})$ is "fully irreducible" if $\Phi^{\omega} = SO(3)$: in this case there are neither gauge symmetries nor gauge copies $[\bar{\mathcal{G}}^{\Omega}_{ROT} = \bar{\mathcal{G}}^{\omega}_{ROT} = Z_3]$ and the holonomy bundle $P^{\omega}(p)$ of every point $p \in \Sigma_\tau \times SO(3)$ coincides with $\Sigma_\tau \times SO(3)$ itself, so that every two points in $\Sigma_\tau \times SO(3)$ can be joined by a ω-horizontal curve. Only in this case the covariant divergence is an elliptic operator without zero modes (this requires the use of special weighted Sobolev spaces for the spin connections to exclude the irreducible and reducible ones) and its Green function can be globally defined (absence of Gribov ambiguities).

In conclusion, the following diagram

$$
\begin{array}{ccccccc}
 & \rightarrow & & {}^3\omega_{r(a)} & \rightarrow & {}^3\Omega_{rs(a)} & \\
{}^3e_{(a)r} & & & & & \Updownarrow & \\
 & \rightarrow & {}^3g_{rs} \;\rightarrow & {}^3\Gamma^{u}_{rs} & \rightarrow & {}^3R^{u}{}_{vrs}, &
\end{array}
\tag{21}
$$

together with Eqs.(16), (18), implies that, to avoid any kind of pathology associated with stability subgroups of gauge transformations, one has to work with cotriads belonging to a function space such that: i) there is no subgroup of isometries in the action of $Diff\, \Sigma_\tau$ on the cotriads (no cone over cone structure of singularities in the lower branch of the diagram); ii) all the spin connections associated with the cotriads are fully irreducible (no type of Gribov ambiguity in the upper branch of the diagram). Both these requirements point towards the use of special weighted Sobolev spaces like in Yang-Mills theory [2,45] (see Appendix C and its bibliography).

It would be useful to make a systematic study of the relationships between the stability groups $\mathcal{S}_R(\Sigma_\tau, {}^3g) \supseteq \mathcal{S}_\Gamma(\Sigma_\tau, {}^3g) \supseteq Iso\,(\Sigma_\tau, {}^3g)$ and the stability groups $\bar{\mathcal{G}}^{\Omega}_{ROT} \supseteq \bar{\mathcal{G}}^{\omega}_{ROT}$ and to show rigorously that the presence of isometries (Gribov ambiguity) in the lower (upper) branch of the diagram implies the existence of Gribov ambiguity (isometries) in the upper (lower) branch.

In Ref. [6] there will be a complete discussion on the definition of proper gauge transformations [Eqs.(6) plus boundary conditions on the parameters of gauge transformations (implying their angle-independent approach to the identity at spatial infinity) will be needed] , problem connected with the differentiability of the Dirac Hamiltonian, with supertranslations, and with the asymptotic behaviour of the constraints and of their gauge parameters. There will be also the definition of the asymptotic Poincaré charges, which are the analogue of the non-Abelian charges (generators of the improper 'global or rigid' gauge transformations) of Yang-Mills theory; see Refs. [46,47] for interpretational problems. Instead, see Ref. [48] for a treatment of large diffeomorphisms, the analogous of the large gauge transformations (due to winding number) of Yang-Mills theory [2], not connected to the identity.

III. MULTITEMPORAL EQUATIONS AND THEIR SOLUTION.

In this Section we study the multitemporal equations associated with the gauge transformations in $\bar{\mathcal{G}}_R$, to find a local parametrization of the cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$ in terms of the parameters $\xi_r(\tau,\vec{\sigma})$ and $\alpha_{(a)}(\tau,\vec{\sigma})$ of $\bar{\mathcal{G}}_R$. We shall assume to have chosen a global coordinate system Ξ on $\Sigma_\tau \approx R^3$ to conform with the discussion of the previous Section.

Let us start with the invariant subalgebra $\bar{g}_{ROT}$ [the algebra of $\bar{\mathcal{G}}_{ROT}$] of rotations, whose generators are the vector fields $X_{(a)}(\tau,\vec{\sigma})$ of Eqs.(11). Since the group manifold of $\bar{\mathcal{G}}_{ROT}$ is a trivial principal bundle $\Sigma_\tau^{(\Xi)} \times SO(3) \approx L\Sigma_\tau$ over Σ_τ, endowed with the coordinate system Ξ, with structure group SO(3) [to be replaced by SU(2) when one studies the action of $\bar{\mathcal{G}}_{ROT}$ on fermion fields], we can use the results of Ref. [2] for the case of SO(3) Yang-Mills theory.

Let $\alpha_{(a)}$ be canonical coordinates of first kind on the group manifold of SO(3). If $r^{(a)}$ are the generators of so(3), $[r^{(a)}, r^{(b)}] = \epsilon_{(a)(b)(c)} r^{(c)}$ [instead $\hat{R}^{(a)}$ are the generators in the adjoint representation, $(\hat{R}^{(a)})_{(b)(c)} = \epsilon_{(a)(b)(c)}$], and if $\gamma_\alpha(s) = exp_{SO(3)}\,(s\alpha_{(a)} r^{(a)})$ is a one-parameter subgroup of SO(3) with tangent vector $\alpha_{(a)} r^{(a)}$ at the identity $I \in SO(3)$, then the element $\gamma_\alpha(1) = exp_{SO(3)}(\alpha_{(a)} r^{(a)}) \in N_I \subset SO(3)$ [N_I is a neighbourhood of the identity such that $exp_{SO(3)}$ is a diffeomorphism from a neighbourhood of $0 \in so(3)$ to N_I] is given coordinates $\{\alpha_{(a)}\}$. If $\tilde{Y}_{(a)}$ and $\tilde{\theta}_{(a)}$ are dual bases ($i_{\tilde{Y}_{(a)}} \tilde{\theta}_{(b)} = \delta_{(a)(b)}$) of left invariant vector fields and left invariant (or Maurer-Cartan) 1-forms on SO(3), we have the standard Maurer-Cartan structure equations

$$[\tilde{Y}_{(a)}, \tilde{Y}_{(b)}] = \epsilon_{(a)(b)(c)} \tilde{Y}_{(c)}$$

$[\tilde{Y}_{(a)}|_I = r^{(a)} \in so(3)]$ and

$$d\tilde{\theta}_{(a)} = -\tfrac{1}{2}\epsilon_{(a)(b)(c)} \tilde{\theta}_{(b)} \wedge \tilde{\theta}_{(c)}$$

$[\tilde{\theta}_{(a)}|_I = r_{(a)} \in so(3)^*$, the dual Lie algebra; $TSO(3) \approx so(3)$, $T^*SO(3) \approx so(3)^*]$. Then, from Lie theorems, in arbitrary coordinates on the group manifold we have

$$\tilde{Y}_{(a)} = B_{(b)(a)}(\alpha) \tfrac{\partial}{\partial \alpha_{(b)}},\ \tilde{\theta}_{(a)} = A_{(a)(b)}(\alpha) d\alpha_{(b)},\ A(\alpha) = B^{-1}(\alpha),\ A(0) = B(0) = 1,$$

and the Maurer-Cartan equations become

$$\tfrac{\partial A_{(a)(c)}(\alpha)}{\partial \alpha_{(b)}} - \tfrac{\partial A_{(a)(b)}(\alpha)}{\partial \alpha_{(c)}} = -\epsilon_{(a)(u)(v)} A_{(u)(b)}(\alpha) A_{(v)(c)}(\alpha),$$

$$\tilde{Y}_{(b)} B_{(a)(c)}(\alpha) - \tilde{Y}_{(c)} B_{(a)(b)}(\alpha) = B_{(u)(b)}(\alpha) \tfrac{\partial B_{(a)(c)}(\alpha)}{\partial \alpha_{(u)}} - B_{(u)(c)}(\alpha) \tfrac{\partial B_{(a)(b)}(\alpha)}{\partial \alpha_{(u)}} = B_{(a)(u)}(\alpha) \epsilon_{(u)(b)(c)}.$$

By definition these coordinates are said canonical of first kind and satisfy $A_{(a)(b)}(\alpha)\,\alpha_{(b)} = \alpha_{(a)}$ [compare with Eq.(A4) of Appendix A], so that we get $A(\alpha) = (e^{R\alpha} - 1)/R\alpha$ with $(R\alpha)_{(a)(b)} = (\hat{R}^{(c)})_{(a)(b)} \alpha_{(c)} = \epsilon_{(a)(b)(c)} \alpha_{(c)}$. The canonical 1-form on SO(3) is $\tilde{\omega}_{SO(3)} = \tilde{\theta}_{(a)} r^{(a)} = A_{(a)(b)}(\alpha) d\alpha_{(b)} r^{(a)}$ [$= a^{-1}(\alpha) d_{SO(3)}\, a(\alpha)$, $a(\alpha) \in SO(3)$; $d_{SO(3)}$ is the exterior derivative on SO(3)]. Due to the Maurer-Cartan structure equations the 1-forms $\tilde{\theta}_{(a)}$ are not integrable on SO(3); however in the neighbourhood $N_I \subset SO(3)$ we can integrate them along the preferred defining line $\gamma_\alpha(s)$ defining the canonical coordinates of first kind to get the phases

$$\Omega^{\gamma_\alpha}_{(a)}(\alpha(s)) = {}_{\gamma_\alpha}\int_I^{\gamma_\alpha(s)}\ \tilde{\theta}_{(a)}|_{\gamma_\alpha} = {}_{\gamma_\alpha}\int_0^{\alpha(s)}\ A_{(a)(b)}(\bar{\alpha})d\bar{\alpha}_{(b)}.$$

If $d_{\gamma_\alpha} = ds \frac{d\alpha_{(a)}(s)}{ds}\frac{\partial}{\partial\alpha_{(a)}}|_{\alpha=\alpha(s)} = d_{SO(3)}|_{\gamma_\alpha(s)}$ is the directional derivative along γ_α, on γ_α we have $d_{\gamma_\alpha}\, \Omega^{\gamma_\alpha}_{(a)}(\alpha(s)) = \tilde{\theta}_{(a)}(\alpha(s))$ and $d_{\gamma_\alpha}\, \tilde{\theta}_{(a)}(\alpha(s)) = 0 \Rightarrow d^2_{\gamma_\alpha} = 0$. The analytic atlas $\mathcal{N}$ for the group manifold of SO(3) is built by starting from the neighbourhood N_I of the identity with canonical coordinates of first kind by left multiplication by elements of SO(3): $\mathcal{N} = \cup_{a\in SO(3)}\, \{a\cdot N_I\}$.

As shown in Ref. [2] for $R^3 \times SO(3)$, in a tubular neighbourhood of the identity cross section σ_I of the trivial principal bundle $R^3 \times SO(3)$ [in which each fiber is a copy of the SO(3) group manifold] we can define generalized canonical coordinates of first kind on each fiber so to build a coordinatization of $R^3 \times SO(3)$. We now extend this construction from the flat Riemannian manifold (R^3, δ_{rs}) to a Riemannian manifold $(\Sigma_\tau, {}^3g_{rs})$ satisfying our hypotheses, especially the Hadamard theorem, so that the 3-manifold Σ_τ, diffeomorphic to R^3, admits global charts.

Let us consider the fiber SO(3) over a point $p \in \Sigma_\tau$, chosen as origin $\vec{\sigma} = 0$ of the global chart Ξ on Σ_τ, with canonical coordinates of first kind $\alpha_{(a)} = \alpha_{(a)}(\tau, \vec{0})$. For a given spin connection ${}^3\omega_{(a)}$ on $\Sigma^{(\Xi)}_\tau \times SO(3)$ let us consider the ${}^3\omega$-horizontal lift of the star of geodesics of the Riemann 3-manifold $(\Sigma_\tau, {}^3g_{rs} = {}^3e_{(a)r}\, {}^3e_{(a)s})$ emanating from $p \in \Sigma_\tau$ through each point of the fiber SO(3). If the spin connection ${}^3\omega_{(a)}$ is fully irreducible, $\Sigma_\tau \times SO(3)$ is in this way foliated by a connection-dependent family of global cross sections defined by the ${}^3\omega$-horizontal lifts of the star of geodesics [they are not ${}^3\omega$-horizontal cross sections, as it was erroneously written in Ref. [2], since such cross sections do not exist when the holonomy groups in each point of $\Sigma_\tau \times SO(3)$ are not trivial]. The canonical coordinates of first kind on the reference SO(3) fiber may then be parallel (with respect to ${}^3\omega_{(a)}$) transported to all the other fibers along these ${}^3\omega$-dependent global cross sections. If $\tilde{p} = (p; \alpha_{(a)}) = (\tau, \vec{0}; \alpha_{(a)}(\tau, \vec{0}))$ is a point in $\Sigma_\tau \times SO(3)$ over $p \in \Sigma_\tau$, if $\sigma_{(\tilde{p})} : \Sigma_\tau \rightarrow \Sigma_\tau \times SO(3)$ is the ${}^3\omega$-dependent cross section through $\tilde{p}$ and if ${}^3\omega^{(\tilde{p})}_{r(a)}(\tau, \vec{\sigma})d\sigma^r = \sigma^*_{(\tilde{p})}\, {}^3\omega_{(a)}$, then the coordinates of the point intersected by $\sigma_{(\tilde{p})}$ on the SO(3) fiber over the point $p^{'}$ of Σ_τ with coordinates $(\tau, \vec{\sigma})$ are

$$\begin{aligned}\alpha_{(a)}(\tau, \vec{\sigma}) &= \alpha_{(b)}(\tau, \vec{0})\, \zeta^{(\omega^{(\tilde{p})}_{(c)})}_{(b)(a)}(\vec{\sigma}, \vec{0}; \tau) = \\ &= \alpha_{(b)}(\tau, \vec{0})\Big(P_{\gamma_{pp^{'}}}\, e^{\int_{\vec{0}}^{\vec{\sigma}} dz^r \hat{R}^{(c)}\, {}^3\omega^{(\tilde{p})}_{r(c)}(\tau, \vec{z})}\Big)_{(b)(a)},\end{aligned} \tag{22}$$

where $\zeta^{(\omega)}_{(b)(a)}(\vec{\sigma}, \vec{0}; \tau)$ is the Wu-Yang nonintegrable phase with the path ordering evaluated along the geodesic $\gamma_{pp^{'}}$ from p to $p^{'}$. The infinitesimal form is

$$\begin{aligned}\alpha_{(a)}(\tau, d\vec{\sigma}) &\approx \alpha_{(a)}(\tau, \vec{0}) + \frac{\partial \alpha_{(a)}(\tau, \vec{\sigma})}{\partial \sigma^r}|_{\vec{\sigma}=0} d\sigma^r \approx \\ &\approx \alpha_{(b)}(\tau, \vec{0})\Big[\delta_{(b)(a)} + (\hat{R}^{(c)})_{(b)(a)}\, {}^3\omega_{r(c)}(\tau, \vec{0})d\sigma^r\Big],\end{aligned} \tag{23}$$

implying that the identity cross section σ_I of $\Sigma^{(\Xi)}_\tau \times SO(3)$ $[\alpha_{(a)} = \alpha_{(a)}(\tau, \vec{0}) = 0]$ is the origin for all SO(3) fibers: $\alpha_{(a)}(\tau, \vec{\sigma})|_{\sigma_I} = 0$. As shown in Ref. [2], on σ_I we also have

$\partial_r \alpha_{(a)}(\tau,\vec{\sigma})|_{\sigma_I} = 0$. The main property of this construction is that these coordinates are such that a vertical infinitesimal increment $d\alpha_{(a)}|_{\alpha=\alpha(\tau,\vec{\sigma})}$ of them along the defining path (one-parameter subgroup) $\gamma_{\alpha(\tau,\vec{\sigma})}(s)$ in the fiber SO(3) over $q \in \Sigma_\tau$ with coordinates $(\tau,\vec{\sigma})$ is numerically equal to the horizontal infinitesimal increment $\partial_r \alpha_{(a)}(\tau,\vec{\sigma}) d\sigma^r$ in going from $\vec{\sigma}$ to $\vec{\sigma}+d\vec{\sigma}$ in Σ_τ

$$d\alpha_{(a)}|_{\alpha=\alpha(\tau,\vec{\sigma})} = d\alpha_{(a)}(\tau,\vec{\sigma}) = \partial_r \alpha_{(a)}(\tau,\vec{\sigma}) d\sigma^r. \tag{24}$$

With this coordinatization of $\Sigma_\tau^{(\Xi)} \times SO(3)$, in the chosen global coordinate system Ξ for Σ_τ in which the identity cross section σ_I is chosen as the origin of the angles, as in Ref. [2] we have the following realization for the vector fields $X_{(a)}(\tau,\vec{\sigma})$ of Eqs.(11)

$$X_{(a)}(\tau,\vec{\sigma}) = B_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) \frac{\tilde{\delta}}{\delta\alpha_{(b)}(\tau,\vec{\sigma})} \Rightarrow \frac{\tilde{\delta}}{\delta\alpha_{(a)}(\tau,\vec{\sigma})} = A_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) X_{(b)}(\tau,\vec{\sigma}), \tag{25}$$

where the functional derivative is the directional functional derivative along the path $\gamma_{\alpha(\tau,\vec{\sigma})}(s)$ in $\Sigma_\tau^{(\Xi)} \times SO(3)$ originating at the identity cross section σ_I (the origin of all SO(3) fibers) in the SO(3) fiber over the point $p \in \Sigma_\tau$ with coordinates $(\tau,\vec{\sigma})$, corresponding in the above construction to the path $\gamma_\alpha(s)$ defining the canonical coordinates of first kind in the reference SO(3) fiber. It satisfies the commutator in Eq.(12) due to the generalized Maurer-Cartan equations for $\Sigma_\tau \times SO(3)$ $[A = B^{-1}]$

$$\begin{aligned}
B_{(u)(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) & \quad \frac{\partial B_{(v)(b)}(\alpha_{(e)})}{\partial \alpha_{(u)}}|_{\alpha=\alpha(\tau,\vec{\sigma})} - B_{(u)(b)}(\alpha_{(e)}(\tau,\vec{\sigma})) \frac{\partial B_{(v)(a)}(\alpha_{(e)})}{\partial \alpha_{(u)}}|_{\alpha=\alpha(\tau,\vec{\sigma})} = \\
&= B_{(v)(d)}(\alpha_{(e)}(\tau,\vec{\sigma})) \epsilon_{(d)(a)(b)}, \\
\frac{\partial A_{(a)(c)}(\alpha_{(e)})}{\partial \alpha_{(b)}}|_{\alpha=\alpha(\tau,\vec{\sigma})} - & \frac{\partial A_{(a)(b)}(\alpha_{(e)})}{\partial \alpha_{(c)}}|_{\alpha=\alpha(\tau,\vec{\sigma})} = \\
&= \epsilon_{(a)(u)(v)} A_{(u)(b)}(\alpha_{(e)}(\tau,\vec{\sigma})) A_{(v)(c)}(\alpha_{(e)}(\tau,\vec{\sigma})),
\end{aligned} \tag{26}$$

holding pointwise on each fiber of $\Sigma_\tau^{(\Xi)} \times SO(3)$ over $(\tau,\vec{\sigma})$ in a suitable tubular neighbourhood of the identity cross section.

By defining a generalized canonical 1-form for $\bar{\mathcal{G}}_{ROT}$,

$\tilde{\omega} = \hat{R}^{(a)}\, \tilde{\theta}_{(a)}(\tau,\vec{\sigma}) = H_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) d\alpha_{(a)}(\tau,\vec{\sigma})$,
where
$\tilde{\theta}_{(a)}(\tau,\vec{\sigma}) = \hat{\theta}_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma}), \partial_r \alpha_{(e)}(\tau,\vec{\sigma})) = \tilde{\theta}_{(a)r}(\tau,\vec{\sigma}) d\sigma^r = A_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma})) d\alpha_{(b)}(\tau,\vec{\sigma})$

are the generalized Maurer-Cartan 1-forms on the Lie algebra $\bar{g}_{ROT}$ of $\bar{\mathcal{G}}_{ROT}$ and where we defined the matrices $H_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) = \hat{R}^{(b)}\, A_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))$, the previous equations can be rewritten in the form of a zero curvature condition

$$\frac{\partial H_{(a)}(\alpha_{(e)})}{\partial \alpha_{(b)}}|_{\alpha=\alpha(\tau,\vec{\sigma})} - \frac{\partial H_{(b)}(\alpha_{(e)})}{\partial \alpha_{(a)}}|_{\alpha=\alpha(\tau,\vec{\sigma})} + [H_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma})), H_{(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))] = 0. \tag{27}$$

Eq.(64) of I and Eqs.(11) and (25) give the following multitemporal equations for the dependence of the cotriad ${}^3e_{(a)r}(\tau,\vec{\sigma})$ on the 3 gauge angles $\alpha_{(a)}(\tau,\vec{\sigma})$

$$
\begin{aligned}
X_{(b)}(\tau,\vec{\sigma}^{'})\,{}^3e_{(a)r}(\tau,\vec{\sigma}) &= B_{(c)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}^{'})\frac{\tilde{\delta}\,{}^3e_{(a)r}(\tau,\vec{\sigma})}{\delta\alpha_{(c)}(\tau,\vec{\sigma}')} =\\
&= -\{{}^3e_{(a)r}(\tau,\vec{\sigma}),{}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}^{'})\} = -\epsilon_{(a)(b)(c)}\,{}^3e_{(c)r}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}),
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad \frac{\tilde{\delta}\,{}^3e_{(a)r}(\tau,\vec{\sigma})}{\delta\alpha_{(b)}(\tau,\vec{\sigma}')} &= -\epsilon_{(a)(c)(d)}A_{(c)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\,{}^3e_{(a)r}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}) =\\
&= \Big[\hat{R}^{(c)}A_{(c)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\Big]_{(a)(d)}\,{}^3e_{(d)r}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}) =\\
&= \Big[H_{(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\Big]_{(a)(d)}\,{}^3e_{(d)r}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}).
\end{aligned}
\tag{28}
$$

These equations are a functional multitemporal generalization of the matrix equation $\frac{d}{dt}U(t,t_o) = h\,U(t,t_o)$, $U(t_o,t_o)=1$, generating the concept of time-ordering. They are integrable (i.e. their solution is path independent) due to Eqs.(27) and their solution is

$$
{}^3e_{(a)r}(\tau,\vec{\sigma}) = {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\,{}^3\bar{e}_{(b)r}(\tau,\vec{\sigma}),
\tag{29}
$$

where [l is an arbitrary path originating at the identity cross section of $\Sigma^{(\Xi)}_\tau\times SO(3)$; due to the path independence it can be replaced with the defining path $\gamma_{(\alpha(\tau,\vec{\sigma})}(s)=\hat{\gamma}(\tau,\vec{\sigma};s)$]

$$
\begin{aligned}
{}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma})) &= \Big(P\,e^{(l)\int_0^{\alpha_{(e)}(\tau,\vec{\sigma})}H_{(c)}(\bar{\alpha}_{(e)})\mathcal{D}\bar{\alpha}_{(c)}}\Big)_{(a)(b)} =\\
&= \Big(P\,e^{(\hat{\gamma})\int_0^{\alpha_{(e)}(\tau,\vec{\sigma})}H_{(c)}(\bar{\alpha}_{(e)})\mathcal{D}\bar{\alpha}_{(c)}}\Big)_{(a)(b)} =\\
&= \Big(P\,e^{\Omega^{\hat{\gamma}}(\alpha_{(e)}(\tau,\vec{\sigma}))}\Big)_{(a)(b)},
\end{aligned}
\tag{30}
$$

is a point dependent rotation matrix [${}^3R^T_{(a)(b)}(\alpha) = {}^3R^{-1}_{(a)(b)}(\alpha)$ since $\hat{R}^{(a)\dagger}=-\hat{R}^{(a)}$].

In Eq.(30) we introduced the generalized phase obtained by functional integration along the defining path in $\Sigma^{(\Xi)}_\tau\times SO(3)$ of the generalized Maurer-Cartan 1-forms

$$
\begin{aligned}
\Omega^{\hat{\gamma}}(\alpha_{(e)}(\tau,\vec{\sigma};s)) &= {}_{(\hat{\gamma})}\int_I^{\gamma_{\alpha(\tau,\vec{\sigma};s)}}\hat{R}^{(a)}\tilde{\theta}_{(a)}|_{\gamma_{\alpha(\tau,\vec{\sigma};s)}} =\\
&= {}_{(\hat{\gamma})}\int_0^{\alpha_{(e)}(\tau,\vec{\sigma};s)}H_{(a)}(\bar{\alpha}_{(e)})\mathcal{D}\bar{\alpha}_{(a)} =\\
&= {}_{(\hat{\gamma})}\int_0^{\alpha_{(e)}(\tau,\vec{\sigma};s)}\hat{R}^{(a)}A_{(a)(b)}(\bar{\alpha}_{(e)})\mathcal{D}\bar{\alpha}_{(b)}.
\end{aligned}
\tag{31}
$$

As shown in Ref. [2], we have

$$
d_{\hat{\gamma}}\,\Omega^{\hat{\gamma}}(\alpha_{(e)}(\tau,\vec{\sigma};s)) = \hat{R}^{(a)}\hat{\theta}_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma};s),\partial_r\alpha_{(e)}(\tau,\vec{\sigma};s)),
$$

where $d_{\hat{\gamma}}$ is the restriction of the "fiber or vertical" derivative d_V on $\Sigma^{(\Xi)}_\tau\times SO(3)$ [the BRST operator] to the defining path, satisfying $d^2_{\hat{\gamma}}=0$ due to the generalized Maurer-Cartan equations.

In Eq.(29), ${}^3\bar{e}_{(a)r}(\tau,\vec{\sigma})$ are the cotriads evaluated at $\alpha_{(a)}(\tau,\vec{\sigma})=0$ (i.e. on the identity cross section). Being Cauchy data of Eqs.(28), they are independent from the angles

$\alpha_{(a)}(\tau,\vec\sigma)$, satisfy $\{{}^3\bar e_{(a)r}(\tau,\vec\sigma), {}^3\tilde M_{(b)}(\tau,\vec\sigma^{'})\}=0$ and depend only on 6 independent functions [the $\alpha_{(a)}(\tau,\vec\sigma)$ are the 3 rotational degrees of freedom hidden in the 9 variables ${}^3e_{(a)r}(\tau,\vec\sigma)$].

We have not found 3 specific conditions on cotriads implying their independency from the angles $\alpha_{(a)}$.

Since from Eq.(34) of I we get for the spin connection [$\hat D^{(\omega)}_{(a)(b)r}(\tau,\vec\sigma)$ is the SO(3) covariant derivative in the adjoint representation]

$$
\begin{aligned}
X_{(b)}(\tau,\vec\sigma^{'})\,{}^3\omega_{r(a)}(\tau,\vec\sigma) &= B_{(c)(b)}(\alpha_{(e)}(\tau,\vec\sigma^{'}))\frac{\tilde\delta\,{}^3\omega_{r(a)}(\tau,\vec\sigma)}{\delta\alpha_{(c)}(\tau,\vec\sigma^{'})}=\\
&=-\{{}^3\omega_{r(a)}(\tau,\vec\sigma), {}^3\tilde M_{(b)}(\tau,\vec\sigma^{'})\}=\\
&=\Big[\delta_{(a)(b)}\partial_r+\epsilon_{(a)(c)(b)}\,{}^3\omega_{r(c)}(\tau,\vec\sigma)\Big]\delta^3(\vec\sigma,\vec\sigma^{'})=\\
&=\Big[\delta_{(a)(b)}\partial_r-(\hat R^{(c)}\,{}^3\omega_{r(c)}(\tau,\vec\sigma))_{(a)(b)}\Big]\delta^3(\vec\sigma,\vec\sigma^{'})=\hat D^{(\omega)}_{(a)(b)r}(\tau,\vec\sigma)\delta^3(\vec\sigma,\vec\sigma^{'}),
\end{aligned}
\tag{32}
$$

which is the same result as for the gauge potential of the SO(3) Yang-Mills theory, we can use the results of Ref. [2] to write the solution of Eq.(32)

$$
\begin{aligned}
{}^3\omega_{r(a)}(\tau,\vec\sigma) &= A_{(a)(b)}(\alpha_{(e)}(\tau,\vec\sigma))\partial_r\alpha_{(b)}(\tau,\vec\sigma)+{}^3\omega^{(T)}_{r(a)}(\tau,\vec\sigma,\alpha_{(e)}(\tau,\vec\sigma)),\\
&\quad with\\
\frac{\partial\,{}^3\omega^{(T)}_{r(a)}(\tau,\vec\sigma,\alpha_{(e)})}{\partial\alpha_{(b)}} &|_{\alpha=\alpha(\tau,\vec\sigma)}=-\epsilon_{(a)(d)(c)}A_{(d)(b)}(\alpha_{(e)}(\tau,\vec\sigma))\,{}^3\omega^{(T)}_{r(c)}(\tau,\vec\sigma,\alpha_{(e)}(\tau,\vec\sigma)).
\end{aligned}
\tag{33}
$$

In ${}^3\omega_{r(a)}(\tau,\vec\sigma)d\sigma^r=\tilde\theta_{(a)}(\tau,\vec\sigma)+{}^3\omega^{(T)}_{r(a)}(\tau,\vec\sigma,\alpha_{(e)}(\tau,\vec\sigma))d\sigma^r$, the first term is a pure gauge spin connection (the BRST ghost), while the second one is the source of the field strength: ${}^3\Omega_{rs(a)}=\partial_r\,{}^3\omega^{(T)}_{s(a)}-\partial_s\,{}^3\omega^{(T)}_{r(a)}-\epsilon_{(a)(b)(c)}\,{}^3\omega^{(T)}_{r(b)}\,{}^3\omega^{(T)}_{s(c)}$. Moreover, the Hodge decomposition theorem [in the functional spaces where the spin connections are fully irreducible] implies that ${}^3\omega^{(\perp)}_{r(a)}(\tau,\vec\sigma)={}^3\omega^{(T)}_{r(a)}(\tau,\vec\sigma,\alpha_{(e)}(\tau,\vec\sigma))$ satisfies ${}^3\nabla^r\,{}^3\omega^{(\perp)}_{r(a)}=0$.

Since we have $X_{(b)}(\tau,\vec\sigma^{'})\,{}^3\omega^{(\perp)}_{r(a)}(\tau,\vec\sigma)=-\epsilon_{(a)(c)(b)}\,{}^3\omega^{(\perp)}_{r(c)}(\tau,\vec\sigma)\delta^3(\vec\sigma,\vec\sigma^{'})$, we get

$$
\begin{aligned}
\frac{\tilde\delta\,{}^3\omega^{(\perp)}_{r(a)}(\tau,\vec\sigma)}{\delta\alpha_{(b)}(\tau,\vec\sigma^{'})} &= [H_{(b)}(\alpha_{(e)}(\tau,\vec\sigma))]_{(a)(c)}\,{}^3\omega^{(\perp)}_{r(c)}(\tau,\vec\sigma)\delta^3(\vec\sigma,\vec\sigma^{'}),\\
\Rightarrow\quad & {}^3\omega^{(\perp)}_{r(a)}(\tau,\vec\sigma)=\Big(P\,e^{\Omega^{\hat\gamma}(\alpha_{(e)}(\tau,\vec\sigma))}\Big)_{(a)(b)}\,{}^3\bar\omega^{(\perp)}_{r(b)}(\tau,\vec\sigma),\\
& {}^3\nabla^r\,{}^3\bar\omega^{(\perp)}_{r(a)}(\tau,\vec\sigma)=0.
\end{aligned}
\tag{34}
$$

The transverse spin connection ${}^3\bar\omega^{(\perp)}_{r(a)}(\tau,\vec\sigma)$ is independent from the gauge angles $\alpha_{(a)}(\tau,\vec\sigma)$ and is the source of the field strength ${}^3\bar\Omega_{rs(a)}=\partial_r\,{}^3\bar\omega^{(\perp)}_{s(a)}-\partial_s\,{}^3\bar\omega^{(\perp)}_{r(a)}-\epsilon_{(a)(b)(c)}\,{}^3\bar\omega^{(\perp)}_{r(b)}\,{}^3\bar\omega^{(\perp)}_{s(c)}$ invariant under the rotation gauge transformations. Clearly, ${}^3\bar\omega^{(\perp)}_{r(a)}$ is built with the reduced cotriads ${}^3\bar e_{(a)r}$.

Let us remark that for ${}^3\omega^F_{r(a)}(\tau,\vec\sigma)d\sigma^r=\tilde\theta_{(a)}(\tau,\vec\sigma)$ we get ${}^3\Omega_{rs(a)}(\tau,\vec\sigma)=0$ and then ${}^3R_{rsuv}=0$: in this case the Riemannian manifold $(\Sigma_\tau, {}^3g_{rs}={}^3e_{(a)r}\,{}^3e_{(a)s})$ becomes the Euclidean manifold $(R^3,\, {}^3g^F_{rs})$ with ${}^3g^F_{rs}$ the flat 3-metric in curvilinear coordinates. Now

Eq.(29) implies that ${}^3g_{rs} = {}^3\bar{e}_{(a)r}\, {}^3\bar{e}_{(a)s}$ and then ${}^3g^F_{rs}(\tau,\vec{\sigma}) = \frac{\partial\tilde{\sigma}^u}{\partial\sigma^r}\frac{\partial\tilde{\sigma}^v}{\partial\sigma^s}\delta_{uv} = {}^3g^F_{rs}(\vec{\sigma})$, if $\tilde{\sigma}^u(\vec{\sigma})$ are Cartesian coordinates, so that

${}^3\Gamma^{F\,u}_{rs} = \frac{\partial\sigma^u}{\partial\tilde{\sigma}^n}\frac{\partial^2\tilde{\sigma}^n}{\partial\sigma^r\partial\sigma^s} = {}^3e^u_{(a)}\partial_r\, {}^3e_{(a)s} = {}^3\triangle^u_{rs}$

(see after Eqs.(21) of I); then, for an arbitrary ${}^3g$ we have ${}^3\Gamma^u_{rs} = {}^3\triangle^u_{rs} + {}^3\bar{\Gamma}^u_{rs}$ with ${}^3\bar{\Gamma}^u_{rs} = {}^3e^u_{(a)}\, {}^3e_{(b)s}\, {}^3\omega_{r(a)(b)}$ the source of the Riemann tensor. This implies that

${}^3\bar{e}^F_{(a)r}(\tau,\vec{\sigma}) = \frac{\partial\tilde{\sigma}^u}{\partial\sigma^r}\, {}^3\tilde{\bar{e}}^F_{(a)u}(\tau,\vec{\tilde{\sigma}}) = \delta_{(a)u}\frac{\partial\tilde{\sigma}^u(\vec{\sigma})}{\partial\sigma^r}$.

Therefore, a flat cotriad on R^3 has the form

$$ {}^3e^F_{(a)r}(\tau,\vec{\sigma}) = {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\delta_{(b)u}\frac{\partial\tilde{\sigma}^u(\vec{\sigma})}{\partial\sigma^r}. \tag{35} $$

Eqs.(64) of I give the multitemporal equations for the momenta

$$ \begin{aligned} X_{(b)}(\tau,\vec{\sigma}^{\,\prime})\, {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) &= B_{(c)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}^{\,\prime})\frac{\tilde{\delta}\, {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})}{\delta\alpha_{(c)}(\tau,\vec{\sigma}^{\prime})} = \\ &= -\epsilon_{(a)(b)(c)}\, {}^3\tilde{\pi}^r_{(c)}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{\,\prime}), \end{aligned} \tag{36} $$

whose solution is [${}^3\tilde{\bar{\pi}}^r_{(a)}(\tau,\vec{\sigma})$ depends only on 6 independent functions]

$$ {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) = {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\, {}^3\tilde{\bar{\pi}}^r_{(b)}(\tau,\vec{\sigma}). \tag{37} $$

With the definition of SO(3) covariant derivative given in Eq.(32), the constraints $\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) \approx 0$ of Eqs.(61) and (62) of I, may be written as

$$ \begin{aligned} \hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) &= -{}^3e^r_{(a)}(\tau,\vec{\sigma})\Big[{}^3\tilde{\Theta}_r + {}^3\omega_{r(b)}\, {}^3\tilde{M}_{(b)}\Big](\tau,\vec{\sigma}) = \\ &= \hat{D}^{(\omega)}_{(a)(b)r}(\tau,\vec{\sigma})\, {}^3\tilde{\pi}^r_{(b)}(\tau,\vec{\sigma}) \approx 0, \end{aligned} \tag{38} $$

so that we have [${}^3\tilde{\pi}^{(T)r}_{(a)}(\tau,\vec{\sigma})$ is a field with zero SO(3) covariant divergence]

$$ \begin{aligned} {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) &= {}^3\tilde{\pi}^{(T)r}_{(a)}(\tau,\vec{\sigma}) - \int d^3\sigma^{\prime}\, \zeta^{(\omega)r}_{(a)(b)}(\vec{\sigma},\vec{\sigma}^{\,\prime};\tau)\, \hat{\mathcal{H}}_{(b)}(\tau,\vec{\sigma}^{\,\prime}), \\ &\hat{D}^{(\omega)}_{(a)(b)r}(\tau,\vec{\sigma})\, {}^3\tilde{\pi}^{(T)r}_{(b)}(\tau,\vec{\sigma}) \equiv 0. \end{aligned} \tag{39} $$

In this equation, we introduced the Green function of the SO(3) covariant divergence, defined by

$$ \hat{D}^{(\omega)}_{(a)(b)r}(\tau,\vec{\sigma})\, \zeta^{(\omega)r}_{(b)(c)}(\vec{\sigma},\vec{\sigma}^{\,\prime};\tau) = -\delta_{(a)(c)}\delta^3(\vec{\sigma},\vec{\sigma}^{\,\prime}). \tag{40} $$

In Ref. [2], this Green function was evaluated for $\Sigma_\tau = R^3$, the flat Euclidean space, by using the Green function $\vec{c}(\vec{\sigma}-\vec{\sigma}^{\,\prime})$ of the flat ordinary divergence [$\triangle = -\vec{\partial}^2_\sigma$] in Cartesian coordinates

$$\vec{c}(\vec{\sigma}-\vec{\sigma}^{'}) = \vec{\partial}_\sigma\, c(\vec{\sigma}-\vec{\sigma}^{'}) = \frac{\vec{\partial}_\sigma}{\triangle}\delta^3(\vec{\sigma}-\vec{\sigma}^{'}) = \frac{\vec{\sigma}-\vec{\sigma}^{'}}{4\pi|\,\vec{\sigma}-\vec{\sigma}^{'}|^3} = \frac{\vec{n}(\vec{\sigma}-\vec{\sigma}^{'})}{4\pi(\vec{\sigma}-\vec{\sigma}^{'})^2},$$

$$\vec{\partial}_\sigma\cdot\vec{c}(\vec{\sigma}-\vec{\sigma}^{'}) = -\delta^3(\vec{\sigma}-\vec{\sigma}^{'}), \tag{41}$$

where $\vec{n}(\vec{\sigma}-\vec{\sigma}^{'})$ is the tangent to the flat geodesic (straight line segment) joining the point of coordinates $\vec{\sigma}$ and $\vec{\sigma}^{'}$, so that $\vec{n}(\vec{\sigma}-\vec{\sigma}^{'})\cdot\vec{\partial}_\sigma$ is the directional derivative along the flat geodesic.

With our special family of Riemannian 3-manifolds $(\Sigma_\tau, {}^3g)$, we would use Eq.(41) in the special global normal chart in which the star of geodesics originating from the reference point p becomes a star of straight lines. In non normal coordinates, the Green function $\vec{c}(\vec{\sigma}-\vec{\sigma}^{'})$ will be replaced with the gradient of the Synge world function [49] or DeWitt geodesic interval bitensor [41] $\sigma_{DW}(\vec{\sigma},\vec{\sigma}^{'})$ [giving the arc length of the geodesic from $\vec{\sigma}$ to $\vec{\sigma}^{'}$] adapted from the Lorentzian M^4 to the Riemannian Σ_τ, i.e.

$$d^r_{\gamma_{pp'}}(\vec{\sigma},\vec{\sigma}^{'}) = \tfrac{1}{3}\sigma^r_{DW}(\vec{\sigma},\vec{\sigma}^{'}) = \tfrac{1}{3}\,{}^3\nabla^r_\sigma\,\sigma_{DW}(\vec{\sigma},\vec{\sigma}^{'}) = \tfrac{1}{3}\partial^r_\sigma\,\sigma_{DW}(\vec{\sigma},\vec{\sigma}^{'})$$

giving in each point $\vec{\sigma}$ the tangent to the geodesic $\gamma_{pp'}$ joining the points p and $p^{'}$ of coordinates $\vec{\sigma}$ and $\vec{\sigma}^{'}$ in the direction from $p^{'}$ to p. Therefore, the Green function is $[\partial_r d^r_{\gamma_{pp'}}(\vec{\sigma},\vec{\sigma}^{'}) = -\delta^3(\vec{\sigma},\vec{\sigma}^{'})$; $d^r_{\gamma_{pp'}}(\vec{\sigma},\vec{\sigma}^{'})\,\partial_r$ is the directional derivative along the geodesic $\gamma_{pp'}$ at p of coordinates $\vec{\sigma}]$

$$\zeta^{(\omega)r}_{(a)(b)}(\vec{\sigma},\vec{\sigma}^{'};\tau) = d^r_{\gamma_{pp'}}(\vec{\sigma},\vec{\sigma}^{'})\Big(P_{\gamma_{pp'}}\, e^{\int^{\vec{\sigma}}_{\vec{\sigma}'} d\sigma^s_1\, \hat{R}^{(c)}\, {}^3\omega_{s(c)}(\tau,\vec{\sigma}_1)}\Big)_{(a)(b)}, \tag{42}$$

with the path ordering done along the geodesic $\gamma_{pp'}$. This path ordering (Wu-Yang nonintegrable phase or geodesic Wilson line) is defined on all $\Sigma_\tau\times SO(3)$ only if the spin connection is fully irreducible; it is just the parallel transporter of Eq.(22).

Eqs.(29) show the dependence of the cotriad on the 3 angles $\alpha_{(a)}(\tau,\vec{\sigma})$, which therefore must be expressible only in terms of the cotriad itself and satisfy $\{\alpha_{(a)}(\tau,\vec{\sigma}),\alpha_{(b)}(\tau,\vec{\sigma}^{'})\} = 0$. They are the rotational gauge variables, canonically conjugate to Abelianized rotation constraints $\tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma})\approx 0$. From Eqs.(25), since the functional derivatives commute, we see that we have [2,50]

$$\tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma}) = {}^3\tilde{M}_{(b)}(\tau,\vec{\sigma})A_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))\approx 0,$$

$$\begin{aligned}
&\{\tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma}),\tilde{\pi}^{\vec{\alpha}}_{(b)}(\tau,\vec{\sigma}^{'})\} = 0,\\
&\{\alpha_{(a)}(\tau,\vec{\sigma}),\tilde{\pi}^{\vec{\alpha}}_{(b)}(\tau,\vec{\sigma}^{'})\} = -A_{(c)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}^{'}))X_{(c)}(\tau,\vec{\sigma}^{'})\alpha_{(a)}(\tau,\vec{\sigma}) =\\
&\qquad = -\delta_{(a)(b)}\delta^3(\vec{\sigma},\vec{\sigma}^{'}).
\end{aligned} \tag{43}$$

The functional equation determining $\alpha_{(a)}(\tau,\vec{\sigma})$ in terms of ${}^3e_{(a)r}(\tau,\vec{\sigma})$ is

$$\begin{aligned}
-\,\delta_{(a)(b)}\delta^3(\vec{\sigma},\vec{\sigma}^{'}) &= \{\alpha_{(a)}(\tau,\vec{\sigma}),{}^3\tilde{M}_{(c)}(\tau,\vec{\sigma}^{'})\}A_{(c)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}^{'}) =\\
&= \epsilon_{(c)(u)(v)}\,{}^3e_{(u)r}(\tau,\vec{\sigma}^{'})\{\alpha_{(a)}(\tau,\vec{\sigma}),{}^3\tilde{\pi}^r_{(v)}(\tau,\vec{\sigma}^{'})\}A_{(c)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}^{'})) =
\end{aligned}$$

$$= \epsilon_{(c)(u)(v)}\, A_{(c)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}^{'}))\, {}^3e_{(u)r}(\tau,\vec{\sigma}^{'}) \frac{\delta \alpha_{(a)}(\tau,\vec{\sigma})}{\delta\, {}^3e_{(v)r}(\tau,\vec{\sigma}')},$$

$$\Rightarrow \qquad \epsilon_{(b)(u)(v)}\; {}^3e_{(u)r}(\tau,\vec{\sigma}^{'}) \frac{\delta \alpha_{(a)}(\tau,\vec{\sigma})}{\delta\, {}^3e_{(v)r}(\tau,\vec{\sigma}')} = -B_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\delta^3(\vec{\sigma},\vec{\sigma}^{'}),$$

$$\epsilon_{(b)(u)(v)}\; {}^3e_{(u)r}(\tau,\vec{\sigma}^{'}) \Big[A_{(a)(c)}(\alpha_{(e)}(\tau,\vec{\sigma})) \frac{\delta \alpha_{(a)}(\tau,\vec{\sigma})}{\delta\, {}^3e_{(v)r}(\tau,\vec{\sigma}')}\Big] = -\delta_{(a)(b)}\delta^3(\vec{\sigma},\vec{\sigma}^{'}),$$

$$\epsilon_{(b)(u)(v)}\; {}^3e_{(u)r}(\tau,\vec{\sigma}^{'}) \frac{\delta \Omega^{\hat{\gamma}}_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))}{\delta\, {}^3e_{(v)r}(\tau,\vec{\sigma}')} = -\delta_{(a)(b)}\delta^3(\vec{\sigma},\vec{\sigma}^{'}),$$

$$\Rightarrow \qquad \epsilon_{(b)(u)(v)} \frac{\delta \Omega^{\hat{\gamma}}_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))}{\delta\, {}^3e_{(v)r}(\tau,\vec{\sigma}')} = -\frac{1}{3}\delta_{(a)(b)}\, {}^3e^r_{(u)}(\tau,\vec{\sigma}^{'})\delta^3(\vec{\sigma},\vec{\sigma}^{'}),$$

$$\frac{\delta \Omega^{\hat{\gamma}}_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))}{\delta\, {}^3e_{(b)r}(\tau,\vec{\sigma}')} = \frac{1}{6}(\hat{R}^{(u)})_{(a)(b)}\; {}^3e^r_{(u)}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}). \qquad (44)$$

This equation is not integrable like the corresponding one in the Yang-Mills case [2]. Having chosen a global coordinate system Ξ on Σ_τ as the conventional origin of pseudodiffeomorphisms, the discussion in Section II allows to define the trivialization $\Sigma^{(\Xi)}_\tau \times SO(3)$ of the coframe bundle $L\Sigma_\tau$. If:
i) $\sigma^{(\Xi)}_I$ is the identity cross section of $\Sigma^{(\Xi)}_\tau \times SO(3)$, corresponding to the coframe ${}^3\theta^I_{(a)} = {}^3e^I_{(a)r} d\sigma^r$ in $L\Sigma_\tau$ [σ^r are the coordinate functions of Ξ];
ii) $\sigma^{(\Xi)}$ is an arbitrary global cross section of $\Sigma^{(\Xi)}_\tau \times SO(3)$, corresponding to a coframe ${}^3\theta_{(a)} = {}^3e_{(a)r} d\sigma^r$ in $L\Sigma_\tau$, in a tubolar neighbourhood of the identity cross section where the generalized canonical coordinates of first kind on the fibers of $\Sigma^{(\Xi)}_\tau \times SO(3)$ (discussed at the beginning of this Section) are defined;
iii) $\sigma^{(\Xi)}(s)$ is the family of global cross sections of $\Sigma^{(\Xi)}_\tau \times SO(3)$ connecting $\Sigma^{(\Xi)}_I = \sigma^{(\Xi)}(s=0)$ and $\Sigma^{(\Xi)} = \sigma^{(\Xi)}(s=1)$ so that on each fiber the point on $\sigma^{(\Xi)}_I$ is connected with the point on $\Sigma^{(\Xi)}$ by the defining path $\hat{\gamma}$ of canonical coordinates of first kind;
then the formal solution of the previous equation is

$$\Omega^{\hat{\gamma}}_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) = \frac{1}{6}\, {}_{\hat{\gamma}}\int_{{}^3e^I_{(a)r}(\tau,\vec{\sigma})}^{{}^3e_{(a)r}(\tau,\vec{\sigma})} (\hat{R}^{(u)})_{(a)(b)}\; {}^3e^r_{(u)}\, \mathcal{D}\, {}^3e_{(b)r}, \qquad (45)$$

where the path integral is made along the path of coframes connecting ${}^3\theta^I_{(a)}$ with ${}^3\theta_{(a)}$ just described. As in Ref. [2] , to get the angles $\alpha_{(a)}(\tau,\vec{\sigma})$ from $\Omega^{\hat{\gamma}}_{(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))$, we essentially have to invert the equation $\Omega^{\hat{\gamma}}_{(a)}(\alpha_{(e)}) = {}_{\hat{\gamma}}\int_0^{\alpha(e)} A_{(a)(b)}(\bar{\alpha})d\bar{\alpha}_{(b)}$ with $A = (e^{R\alpha}-1)/R\alpha$.

Let us now study the multitemporal equations associated with pseudodiffeomorphisms to find the dependence of ${}^3e_{(a)r}(\tau,\vec{\sigma})$ on the parameters $\xi^r(\tau,\vec{\sigma})$. Disregarding momentarily rotations, let us look for a realization of vector fields $\tilde{Y}_r(\tau,\vec{\sigma})$ satisfying the last line of Eqs.(12). If we put

$$\tilde{Y}_r(\tau,\vec{\sigma}) = -\frac{\partial \xi^s(\tau,\vec{\sigma})}{\partial \sigma^r}\frac{\delta}{\delta \xi^s(\tau,\vec{\sigma})}, \qquad (46)$$

we find

$$
\begin{aligned}
[\tilde{Y}_r(\tau,\vec{\sigma}), \tilde{Y}_s(\tau,\vec{\sigma}^{'})] &= [\frac{\partial \xi^u(\tau,\vec{\sigma})}{\partial \sigma^r}\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma})}, \frac{\partial \xi^v(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}}\frac{\delta}{\delta \xi^v(\tau,\vec{\sigma}^{'})}] = \\
&= \frac{\partial \xi^u(\tau,\vec{\sigma})}{\partial \sigma^r}\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'s}}\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma}^{'})} - \frac{\partial \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}}\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^r}\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma})} = \\
&= -\frac{\partial \xi^u(\tau,\vec{\sigma})}{\partial \sigma^r}\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{s}}\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma}^{'})} + \frac{\partial \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}}\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'r}}\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma})} = \\
&= \Big[-\frac{\partial}{\partial \sigma^s}\Big(\frac{\partial \xi^u(\tau,\vec{\sigma})}{\partial \sigma^r}\delta^3(\vec{\sigma},\vec{\sigma}^{'})\Big) + \frac{\partial^2 \xi^u(\tau,\vec{\sigma})}{\partial \sigma^r \partial \sigma^s}\delta^3(\vec{\sigma},\vec{\sigma}^{'})\Big]\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma}^{'})} + \\
&+ \Big[\frac{\partial}{\partial \sigma^{'r}}\Big(\frac{\partial \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}}\delta^3(\vec{\sigma},\vec{\sigma}^{'})\Big) - \frac{\partial^2 \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'r} \partial \sigma^{'s}}\delta^3(\vec{\sigma},\vec{\sigma}^{'})\Big]\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma})} = \\
&= -\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^s}\frac{\partial \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'r}}\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma}^{'})} + \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'r}}\frac{\partial \xi^u(\tau,\vec{\sigma})}{\partial \sigma^{s}}\frac{\delta}{\delta \xi^u(\tau,\vec{\sigma})} = \\
&= -\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'s}}\tilde{Y}_r(\tau,\vec{\sigma}^{'}) - \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'r}}\tilde{Y}_s(\tau,\vec{\sigma}),
\end{aligned}
\tag{47}
$$

in accord with the last of Eqs.(12). Therefore, the role of the Maurer-Cartan matrix B for rotations is taken by minus the Jacobian matrix of the pseudodiffeomorphism $\vec{\sigma} \mapsto \vec{\xi}(\vec{\sigma})$. To take into account the noncommutativity of rotations and pseudodiffeomorphisms [the second line of Eqs.(12)], we need the definition

$$
Y_r(\tau,\vec{\sigma}) = -\{., {}^3\tilde{\Theta}_r(\tau,\vec{\sigma})\} = -\frac{\partial \xi^s(\tau,\vec{\sigma})}{\partial \sigma^r}\frac{\delta}{\delta \xi^s(\tau,\vec{\sigma})} - \frac{\partial \alpha_{(a)}(\tau,\vec{\sigma})}{\partial \sigma^r}\frac{\tilde{\delta}}{\delta \alpha_{(a)}(\tau,\vec{\sigma})}. \tag{48}
$$

Clearly the last line of Eqs.(12) is satisfied, while regarding the second line we have consistently

$$
\begin{aligned}
[X_{(a)}(\tau,\vec{\sigma}), Y_r(\tau,\vec{\sigma}^{'})] &= -[B_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))\frac{\tilde{\delta}}{\delta \alpha_{(b)}(\tau,\vec{\sigma})}, \frac{\partial \alpha_{(c)}(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'r}}\frac{\tilde{\delta}}{\delta \alpha_{(c)}(\tau,\vec{\sigma}^{'})}] = \\
&= -B_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'r}}\frac{\tilde{\delta}}{\delta \alpha_{(b)}(\tau,\vec{\sigma}^{'})} + \\
&+ \frac{\partial \alpha_{(c)}(\tau,\vec{\sigma})}{\partial \sigma^r}\delta^3(\vec{\sigma},\vec{\sigma}^{'})\frac{\partial B_{(b)(a)}(\alpha_{(e)})}{\partial \alpha_{(c)}}|_{\alpha=\alpha(\tau,\vec{\sigma})}\frac{\tilde{\delta}}{\delta \alpha_{(b)}(\tau,\vec{\sigma})} = \\
&= -\frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'r}}X_{(a)}(\tau,\vec{\sigma}^{'}).
\end{aligned}
\tag{49}
$$

From Eqs.(48) and (25) we get

$$
\begin{aligned}
\frac{\delta}{\delta \xi^r(\tau,\vec{\sigma})} &= -\frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\Big[Y_s(\tau,\vec{\sigma}) + A_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\frac{\partial \alpha_{(b)}(\tau,\vec{\sigma})}{\partial \sigma^s}X_{(a)}(\tau,\vec{\sigma})\Big] = \\
&= \frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\Big[\{., {}^3\tilde{\Theta}_s(\tau,\vec{\sigma})\} + \tilde{\theta}_{(a)s}(\tau,\vec{\sigma})\{., {}^3\tilde{M}_{(a)}(\tau,\vec{\sigma})\}\Big] \stackrel{def}{=} \\
&\stackrel{def}{=} \{., \tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma})\},
\end{aligned}
$$

$$\Rightarrow \quad \{\xi^r(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{\xi}}_s(\tau,\vec{\sigma}^{'})\} = \delta^r_s \delta^3(\vec{\sigma},\vec{\sigma}^{'}),$$
$$\{\tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{\xi}}_s(\tau,\vec{\sigma}^{'})\} = 0, \tag{50}$$

where $\tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma})$ is the momentum conjugate to the 3 gauge variables $\xi^r(\tau,\vec{\sigma})$, which will be functions only of the cotriads. On the space of cotriads the Abelianized form of the pseudodiffeomorphism constraints is

$$\tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma}) = \frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \Big[{}^3\tilde{\Theta}_s(\tau,\vec{\sigma}) + \hat{\tilde{\theta}}_{(a)s}(\alpha_{(e)}(\tau,\vec{\sigma}), \partial_u\alpha_{(e)}(\tau,\vec{\sigma}))\, {}^3\tilde{M}_{(a)}(\tau,\vec{\sigma})\Big] =$$
$$= \frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \Big[{}^3\tilde{\Theta}_s + \frac{\partial \alpha_{(a)}}{\partial \sigma^s} \tilde{\pi}^{\vec{\alpha}}_{(a)}\Big](\tau,\vec{\sigma}) \approx 0, \tag{51}$$

and both $\xi^r(\tau,\vec{\sigma})$ and $\tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma})$ have zero Poisson bracket with $\alpha_{(a)}(\tau,\vec{\sigma})$, $\tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma})$.

Therefore, the 6 gauge variables $\xi^r(\tau,\vec{\sigma})$ and $\alpha_{(a)}(\tau,\vec{\sigma})$ and their conjugate momenta form 6 canonical pairs of a new canonical basis adapted to the rotation and pseudodiffeomorphisms constraints and replacing 6 of the 9 conjugate pairs ${}^3e_{(a)r}(\tau,\vec{\sigma})$, ${}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})$.

From Eqs.(64) of I and from Eqs.(48) and (29), we get

$$\begin{aligned}
Y_s(\tau,\vec{\sigma}^{'})\, {}^3e_{(a)r}(\tau,\vec{\sigma}) &= -\Big(\frac{\partial \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}} \frac{\delta}{\delta \xi^u(\tau,\vec{\sigma}^{'})} + \frac{\partial \alpha_{(c)}(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'r}} \frac{\tilde{\delta}}{\delta \alpha_{(c)}(\tau,\vec{\sigma}^{'})}\Big) \cdot \\
&\cdot \Big[{}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\, {}^3\bar{e}_{(b)r}(\tau,\vec{\sigma})\Big] = \\
&= -{}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma})) \frac{\partial \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}} \frac{\delta\, {}^3\bar{e}_{(b)r}(\tau,\vec{\sigma})}{\delta \xi^u(\tau,\vec{\sigma}^{'})} - \\
&- \frac{\partial \alpha_{(c)}(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}} \frac{\tilde{\delta}^3 R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))}{\delta \alpha_{(c)}(\tau,\vec{\sigma}^{'})}\, {}^3\bar{e}_{(b)r}(\tau,\vec{\sigma}) = \\
&= -{}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma})) \frac{\partial \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}} \frac{\delta\, {}^3\bar{e}_{(b)r}(\tau,\vec{\sigma})}{\delta \xi^u(\tau,\vec{\sigma}^{'})} - \\
&- \frac{\partial\, {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))}{\partial \sigma^s} \delta^3(\vec{\sigma},\vec{\sigma}^{'})\, {}^3\bar{e}_{(b)r}(\tau,\vec{\sigma}) = \\
&= -\{{}^3e_{(a)r}(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_s(\tau,\vec{\sigma}^{'})\} = \\
&= -\frac{\partial\, {}^3e_{(a)r}(\tau,\vec{\sigma})}{\partial \sigma^s} \delta^3(\vec{\sigma},\vec{\sigma}^{'}) + {}^3e_{(a)s}(\tau,\vec{\sigma}) \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^r} = \\
&= -{}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma})) \frac{\partial\, {}^3\bar{e}_{(b)r}(\tau,\vec{\sigma})}{\partial \sigma^s} \delta^3(\vec{\sigma},\vec{\sigma}^{'}) - \\
&- \frac{\partial^3 R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))}{\partial \sigma^s}\, {}^3\bar{e}_{(b)r}(\tau,\vec{\sigma}) \delta^3(\vec{\sigma},\vec{\sigma}^{'}) + \\
&+ {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\, {}^3\bar{e}_{(b)r}(\tau,\vec{\sigma}) \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^r},
\end{aligned} \tag{52}$$

so that the pseudodiffeomorphism multitemporal equations for ${}^3\bar{e}_{(a)r}(\tau,\vec{\sigma})$ are

$$-\tilde{Y}_s(\tau,\vec{\sigma}^{'})\, {}^3\bar{e}_{(a)r}(\tau,\vec{\sigma}) = \frac{\partial \xi^u(\tau,\vec{\sigma}^{'})}{\partial \sigma^{'s}} \frac{\delta\, {}^3\bar{e}_{(a)r}(\tau,\vec{\sigma})}{\delta \xi^u(\tau,\vec{\sigma}^{'})} =$$

$$= \frac{\partial\, {}^3\bar{e}_{(a)r}(\tau,\vec{\sigma})}{\partial\sigma^s}\delta^3(\vec{\sigma},\vec{\sigma}^{'}) - {}^3\bar{e}_{(a)s}(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma'^r}. \tag{53}$$

Analogously, from Eqs.(64) of I and Eqs.(46) and (37) we have

$$\begin{aligned}
Y_s(\tau,\vec{\sigma}^{'})\,{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) &= -\Big(\frac{\partial\xi^u(\tau,\vec{\sigma}^{'})}{\partial\sigma'^s}\frac{\delta}{\delta\xi^u(\tau,\vec{\sigma}^{'})} + \frac{\partial\alpha_{(c)}(\tau,\vec{\sigma}^{'})}{\partial\sigma'^r}\frac{\tilde{\delta}}{\delta\alpha_{(c)}(\tau,\vec{\sigma}^{'})}\Big)\cdot\\
&\cdot\ \Big[{}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\,{}^3\bar{\tilde{\pi}}^r_{(b)}(\tau,\vec{\sigma})\Big] =\\
&= -\Big[{}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}^{'}))\,{}^3\bar{\tilde{\pi}}^r_{(b)}(\tau,\vec{\sigma}^{'})\Big]\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma'^s} +\\
&+ \delta^r_s\,{}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\,{}^3\bar{\tilde{\pi}}^u_{(b)}(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma'^u},
\end{aligned} \tag{54}$$

and we get the pseudodiffeomorphism multitemporal equation for ${}^3\bar{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma})$

$$\begin{aligned}
-\tilde{Y}_s(\tau,\vec{\sigma}^{'})\,{}^3\bar{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}) &= \frac{\partial\xi^u(\tau,\vec{\sigma}^{'})}{\partial\sigma'^s}\frac{\delta\,{}^3\bar{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma})}{\delta\xi^u(\tau,\vec{\sigma}^{'})} =\\
&= -{}^3\bar{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}^{'})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma'^s} - \delta^r_s\,{}^3\bar{\tilde{\pi}}^u_{(a)}(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma'^u}.
\end{aligned} \tag{55}$$

Let us remark that the Jacobian matrix satisfies an equation like (53)

$$\begin{aligned}
-\tilde{Y}_s(\tau,\vec{\sigma}^{'})\quad \frac{\partial\xi^u(\tau,\vec{\sigma})}{\partial\sigma^r} &= \frac{\partial\xi^v(\tau,\vec{\sigma}^{'})}{\partial\sigma'^s}\frac{\delta}{\delta\xi^v(\tau,\vec{\sigma}^{'})}\frac{\partial\xi^u(\tau,\vec{\sigma})}{\partial\sigma^r} =\\
&= \frac{\partial\xi^u(\tau,\vec{\sigma}^{'})}{\partial\sigma'^s}\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r} = -\frac{\partial^u(\tau,\vec{\sigma}^{'})}{\partial\sigma'^s}\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma'^r} =\\
&= -\frac{\partial\xi^u(\tau,\vec{\sigma})}{\partial\sigma^s}\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma'^r} + \frac{\partial^2\xi^u(\tau,\vec{\sigma})}{\partial\sigma^r\partial\sigma^s}\delta^3(\vec{\sigma},\vec{\sigma}^{'}) =\\
&= \frac{\partial}{\partial\sigma^s}(\frac{\partial\xi^u(\tau,\vec{\sigma})}{\partial\sigma^r})\delta^3(\vec{\sigma},\vec{\sigma}^{'}) - \frac{\partial\xi^u(\tau,\vec{\sigma})}{\partial\sigma^s}\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma'^r}.
\end{aligned} \tag{56}$$

so that the identity $\frac{\partial\xi^u(\tau,\vec{\sigma}^{'})}{\partial\sigma'^s}\frac{\delta f(\tau,\vec{\xi}(\tau,\vec{\sigma}))}{\delta\xi^u(\tau,\vec{\sigma}^{'})} = \frac{\partial f(\tau,\vec{\xi}(\tau,\vec{\sigma}))}{\partial\sigma^s}\delta^3(\vec{\sigma},\vec{\sigma}^{'})$, implies the following solutions of the multitemporal equations [again $\hat{V}(\vec{\xi}(\tau,\vec{\sigma}))$ is the operator with the action $\hat{V}(\vec{\xi}(\tau,\vec{\sigma}))f(\tau,\vec{\sigma}) = f(\tau,\vec{\xi}(\tau,\vec{\sigma}))$; and Eqs.(13) is used]

$$\begin{aligned}
{}^3\bar{e}_{(a)r}(\tau,\vec{\sigma}) &= \frac{\partial\xi^s(\tau,\vec{\sigma})}{\partial\sigma^r}\,{}^3\hat{e}_{(a)s}(\tau,\vec{\xi}(\tau,\vec{\sigma})) = \frac{\partial\xi^s(\tau,\vec{\sigma})}{\partial\sigma^r}\hat{V}(\vec{\xi}(\tau,\vec{\sigma}))\,{}^3\hat{e}_{(a)s}(\tau,\vec{\sigma}),\\
&\frac{\delta\,{}^3\hat{e}_{(a)r}(\tau,\vec{\sigma})}{\delta\xi^s(\tau,\vec{\sigma}^{'})} = 0,\\
{}^3e_{(a)r}(\tau,\vec{\sigma}) &= {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\frac{\partial\xi^s(\tau,\vec{\sigma})}{\partial\sigma^r}\,{}^3\hat{e}_{(b)s}(\tau,\vec{\xi}(\tau,\vec{\sigma})) =\\
&= \frac{\partial\xi^s(\tau,\vec{\sigma})}{\partial\sigma^r}\,{}^3R_{(a)(b)}(\alpha^{'}_{(e)}(\tau,\vec{\xi}(\tau,\vec{\sigma})))\,{}^3\hat{e}_{(b)s}(\tau,\vec{\xi}(\tau,\vec{\sigma})),\\
{}^3g_{rs}(\tau,\vec{\sigma}) &= \frac{\partial\xi^u(\tau,\vec{\sigma})}{\partial\sigma^r}\frac{\partial\xi^v(\tau,\vec{\sigma})}{\partial\sigma^s}\,{}^3\hat{e}_{(a)u}(\tau,\vec{\xi}(\tau,\vec{\sigma}))\,{}^3\hat{e}_{(a)v}(\tau,\vec{\xi}(\tau,\vec{\sigma})).
\end{aligned} \tag{57}$$

Here the cotriads ${}^3\hat{e}_{(a)r}(\tau,\vec{\sigma})$ depend only on 3 degrees of freedom and are Dirac observables with respect to both Abelianized rotations and pseudodiffeomorphisms. Again, like in the case of rotations, we have not found 3 specific conditions on the cotriads implying this final reduction. This is due to the fact that, even if one has a trivial coframe bundle, one does not know the group manifold of $Diff\,\Sigma_\tau$ and that there is no canonical identity for pseudodiffeomorphisms and therefore also for rotations inside the gauge group $\bar{\mathcal{G}}_R$.

Eqs.(57) are the counterpart in tetrad gravity of the solutions of the 3 elliptic equations for the gravitomagnetic vector potential $\check{W}^r$ of the conformal approach (see the end of Appendix C).

If $\frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^s}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}$ is the inverse Jacobian matrix and $|\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}|$ the determinant of the Jacobian matrix, the following identities

$$
\begin{aligned}
\delta^r_s &= \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial \xi^u(\tau,\vec{\sigma})}{\partial \sigma^s},\\
\Rightarrow &\quad \frac{\partial}{\partial \sigma^v}\frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} = -\frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^w}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial^2 \xi^w(\tau,\vec{\sigma})}{\partial \sigma^v \partial \sigma^s},\\
\Rightarrow &\quad \frac{\delta}{\delta \xi^v(\tau,\vec{\sigma}^{'})}\frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} = -\frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^s}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^v},\\
&\quad \downarrow\\
-\tilde{Y}_s(\tau,\vec{\sigma}^{'}) &\quad \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} =\\
&= \frac{\partial}{\partial \sigma^s}\Big(\frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\Big)\delta^3(\vec{\sigma},\vec{\sigma}^{'}) + \delta^r_s \frac{\partial \sigma^v(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}-\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'v}},
\end{aligned}
\tag{58}
$$

and [use is done of $\delta ln\, det\, M = Tr\,(M^{-1}\delta M)$]

$$
\begin{aligned}
\frac{\partial}{\partial \sigma^r}|\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}| &= |\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}| \frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial^2 \xi^u(\tau,\vec{\sigma})}{\partial \sigma^r \partial \sigma^s},\\
\frac{\delta}{\delta \xi^r(\tau,\vec{\sigma}^{'})}|\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}| &= |\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}| \frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^s},\\
&\downarrow\\
-\tilde{Y}_s(\tau,\vec{\sigma}^{'})|\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}| &= -|\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}| \frac{\partial \delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial \sigma^{'s}},
\end{aligned}
\tag{59}
$$

allow to get

$$
\begin{aligned}
{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) &= {}^3R_{(a)(b)}(\alpha_{(a)}(\tau,\vec{\sigma}))\, {}^3\bar{\tilde{\pi}}^r_{(b)}(\tau,\vec{\sigma}) =\\
&= {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\, |\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}|\, \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^s}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\, {}^3\hat{\tilde{\pi}}^s_{(b)}(\tau,\vec{\xi}(\tau,\vec{\sigma})) =\\
&= {}^3R_{(a)(b)}(\alpha_{(e)}(\tau.\vec{\sigma}))\, |\frac{\partial \xi(\tau,\vec{\sigma})}{\partial \sigma}|\, \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^s}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\, \hat{V}(\vec{\xi}(\tau,\vec{\sigma}))\, {}^3\hat{\tilde{\pi}}^s_{(b)}(\tau.\vec{\sigma}),
\end{aligned}
\tag{60}
$$

where ${}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma})$ are Dirac observables with respect to both Abelianized rotations and pseudodiffeomorphisms. In a similar way we get

$$
{}^3e^r_{(a)}(\tau,\vec \sigma) = {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec \sigma)) \frac{\partial \sigma^r(\vec \xi)}{\partial \xi^s}|_{\vec \xi = \vec \xi (\tau,\vec \sigma)}\, {}^3\hat e^r_{(b)}(\tau, \vec \xi (\tau,\vec \sigma)), \tag{61}
$$

with ${}^3\hat e^r_{(a)}(\tau,\vec \sigma)$ the Dirac observables for triads dual to ${}^3e_{(a)r}(\tau,\vec \sigma)$. The line element becomes

$$
\begin{aligned}
ds^2 &= \epsilon \Big([N_{(as)}+n]^2 - [N_{(as)r}+n_r] \frac{\partial \sigma^r(\vec \xi)}{\partial \xi^u} {}^3\hat e^u_{(a)}(\vec \xi)\, {}^3\hat e^v_{(a)}(\vec \xi) \frac{\partial \sigma^s(\vec \xi)}{\partial \xi^v} [N_{(as)s}+n_s] \Big) (d\tau)^2 - \\
&- 2\epsilon [N_{(as)r}+n_r] d\tau d\sigma^r - \epsilon \frac{\partial \xi^u}{\partial \sigma^r}\, {}^3\hat e_{(a)u}(\vec \xi)\, {}^3\hat e_{(a)v}(\vec \xi) \frac{\partial \xi^v}{\partial \sigma^s} d\sigma^r d\sigma^s = \\
&= \epsilon \Big([N_{(as)}+n]^2 (d\tau)^2 - [{}^3\hat e_{(a)u}(\vec \xi) \frac{\partial \xi^u}{\partial \sigma^r} d\sigma^r + {}^3\hat e^u_{(a)}(\vec \xi) \frac{\partial \sigma^r(\vec \xi)}{\partial \xi^u} (N_{(as)r}+n_r) d\tau] \\
&\quad [{}^3\hat e_{(a)v}(\vec \xi) \frac{\partial \xi^v}{\partial \sigma^s} d\sigma^s + {}^3\hat e^v_{(a)}(\vec \xi) \frac{\partial \sigma^s(\vec \xi)}{\partial \xi^v} (N_{(as)s}+n_s) d\tau] \Big) .
\end{aligned} \tag{62}
$$

To get $\xi^r(\tau,\vec \sigma)$ in terms of the cotriads we have to solve the equations [use is done of Eq.(50), of (62) of I and of $\{ \xi^r(\tau,\vec \sigma), {}^3\tilde M_{(a)}(\tau,{\vec \sigma}^{'}) \} =0$]

$$
\begin{aligned}
\delta^r_s \delta^3(\vec \sigma ,{\vec \sigma}^{'}) &= \{ \xi^r(\tau,\vec \sigma), {\tilde \pi}^{\vec \xi}_s(\tau,{\vec \sigma}^{'}) \} = \\
&= \frac{\partial \sigma^u(\vec \xi)}{\partial \xi^s}|_{\vec \xi = \vec \xi (\tau,{\vec \sigma}^{'})}\, \{ \xi^r(\tau,\vec \sigma), {}^3\tilde \Theta_u(\tau,{\vec \sigma}^{'}) \} = \\
&= \frac{\partial \sigma^u(\vec \xi)}{\partial \xi^s}|_{\vec \xi = \vec \xi (\tau,{\vec \sigma}^{'})} \Big[\Big(\frac{\partial\, {}^3e_{(a)v}(\tau,{\vec \sigma}^{'})}{\partial \sigma^{'u}} - \frac{\partial\, {}^3e_{(a)u}(\tau,{\vec \sigma}^{'})}{\partial \sigma^{'v}} \Big) \frac{\delta \xi^r(\tau,\vec \sigma)}{\delta\, {}^3e_{(a)v}(\tau,{\vec \sigma}^{'})} - \\
&- {}^3e_{(a)u}(\tau,{\vec \sigma}^{'}) \frac{\partial}{\partial \sigma^{'v}} \frac{\delta \xi^r(\tau,\vec \sigma)}{\delta\, {}^3e_{(a)v}(\tau,{\vec \sigma}^{'})} \Big] ,\\
&\Downarrow \\
\Big(\Big[\delta_{(a)(b)} \partial^{'}_v &- {}^3e^u_{(a)} (\partial^{'}_u\, {}^3e_{(b)v} - \partial^{'}_v\, {}^3e_{(b)u}) \Big] (\tau,{\vec \sigma}^{'}) \frac{\delta}{\delta\, {}^3e_{(b)v}(\tau,{\vec \sigma}^{'})} + \\
&+ \delta^3(\vec \sigma ,{\vec \sigma}^{'}) {}^3e^u_{(a)}(\tau,\vec \sigma) \frac{\partial}{\partial \sigma^u} \Big) \xi^r(\tau,\vec \sigma) = 0.
\end{aligned} \tag{63}
$$

We do not know how to solve these equations along some privileged path in the group manifold of $Diff\, \Sigma_{\tau}$ after having chosen a global coordinate system Ξ as a conventional origin of pseudodiffeomorphisms [this identifies a conventional identity cross section $\Sigma^{(\Xi)}_{\tau}$ in the proposed description of $Diff\, \Sigma_{\tau}$ with the fibration $\Sigma_{\tau}\times \Sigma_{\tau} \rightarrow \Sigma_{\tau}$ for the case $\Sigma_{\tau}\approx R^3$], due to the poor understanding of the geometry and differential structure of this group manifold. Presumably, since the fibers of $\Sigma_{\tau}\times \Sigma_{\tau}$ are also copies of Σ_{τ}, on each one of them one can try to define an analogue of canonical coordinates of first kind by using the geodesic exponential map:

i) choose a reference fiber $\Sigma_{\tau ,0}$ in $\Sigma_{\tau}\times \Sigma_{\tau}$ over a point $p=(\tau ,\vec 0)$ chosen as origin in the base (and then connected to all the points in base with geodesics; for $\Sigma_{\tau}\approx R^3$ this is well defined; the global cross sections corresponding to global coordinate systems should be horizontal lifts of this geodesic star with respect to some notion of connection on the fibration);

ii) if q_o is the point in $\Sigma_{\tau}\times \Sigma_{\tau}$ at the intersection of $\Sigma_{\tau ,0}$ with the conventional identity cross section $\Sigma^{(\Xi)}_{\tau}$ and q_1 the point where $\Sigma_{\tau ,0}$ intersects a nearby global cross section $\Sigma^{(\Xi^{'})}_{\tau}$ [$\Xi^{'}$ is

another global coordinate system on Σ_τ], we can consider the geodesic $\gamma_{q_oq_1}$ on $\Sigma_{\tau,0}$;
iii) use the geodesic exponential map along the geodesic $\gamma_{q_oq_1}$ to define "pseudodiffeomorphism coordinates $\vec{\xi}(\tau,\vec{0})$" describing the transition from the global coordinate system Ξ to Ξ' over the base point $p=(\tau,\vec{0})$;
iv) parallel transport these coordinates on the fiber $\Sigma_{\tau,0}$ to the other fibers along the geodesics of the cross sections $\Sigma_\tau^{(\Xi')}$.

If this coordinatization of the group manifold of $Diff\,\Sigma_\tau$ for $\Sigma_\tau\approx R^3$ can be justified, then one could try to solve the previous equations.

Instead, we are able to give a formal expression for the operator $\hat{V}(\vec{\xi}(\vec{\sigma}))$ [for the sake of simplicity we do not consider the τ-dependence], whose action on functions $f(\vec{\sigma})$ is $\hat{V}(\vec{\xi}(\vec{\sigma}))f(\vec{\sigma})=f(\vec{\xi}(\vec{\sigma}))$. We have

$$\hat{V}(\vec{\xi}(\vec{\sigma}))=P_\gamma\, e^{(\int_{\vec{\sigma}}^{\vec{\xi}(\vec{\sigma})}\frac{\partial\sigma^r(u)}{\partial u^s}\mathcal{D}u^s)\frac{\partial}{\partial\sigma^r}}, \tag{64}$$

where the path ordering is along the geodesic γ in Σ_τ joining the points with coordinates $\vec{\sigma}$ and $\vec{\sigma}'=\vec{\xi}(\vec{\sigma})$. For infinitesimal pseudodiffeomorphisms $\vec{\sigma}\mapsto\vec{\sigma}'(\vec{\sigma})=\vec{\xi}(\vec{\sigma})=\vec{\sigma}+\delta\vec{\sigma}(\vec{\sigma})$ [with inverse $\vec{\sigma}'=\vec{\xi}\mapsto\vec{\sigma}(\vec{\xi})=\vec{\xi}-\delta\vec{\sigma}(\vec{\xi})$], we have

$$\begin{aligned}\hat{V}(\vec{\sigma}+\delta\vec{\sigma})&\approx 1+\Big[\delta\sigma^s(\vec{\sigma})\frac{\partial\sigma^r(\vec{\xi})}{\partial\xi^s}|_{\vec{\xi}(\vec{\sigma})-\delta\vec{\sigma}(\vec{\xi}(\vec{\sigma}))}\Big]\frac{\partial}{\partial\sigma^r}\approx 1+\delta\sigma^s(\vec{\sigma})\frac{\partial}{\partial\sigma^s}:\\ &:\ f(\vec{\sigma})\mapsto f(\vec{\sigma})+\delta\sigma^s(\vec{\sigma})\frac{\partial f(\vec{\sigma})}{\partial\sigma^s}\approx f(\vec{\sigma}+\delta\vec{\sigma}(\vec{\sigma})).\end{aligned} \tag{65}$$

Formally we have [if $\delta/\delta\xi^r(\vec{\sigma})$ is interpreted as the directional functional derivative along γ]

$$\begin{aligned}\frac{\delta}{\delta\xi^r(\vec{\sigma}')}[\hat{V}(\vec{\xi}(\vec{\sigma}))f(\vec{\sigma})]&=\delta^3(\vec{\sigma},\vec{\sigma}')\frac{\partial\sigma^s(\vec{\xi})}{\partial\xi^r}|_{\vec{\xi}=\vec{\xi}(\vec{\sigma})}\frac{\partial}{\partial\sigma^s}[\hat{V}(\vec{\xi}(\vec{\sigma}))f(\vec{\sigma})]=\\ &=\delta^3(\vec{\sigma},\vec{\sigma}')\frac{\partial\sigma^s(\vec{\xi})}{\partial\xi^r}|_{\vec{\xi}=\vec{\xi}(\vec{\sigma})}\frac{\partial f(\vec{\xi}(\vec{\sigma}))}{\partial\sigma^s}=\delta^3(\vec{\sigma},\vec{\sigma}')\frac{\partial f(\vec{\xi})}{\partial\xi^r}|_{\vec{\xi}=\vec{\xi}(\vec{\sigma})}=\\ &=\frac{\delta f(\vec{\xi}(\vec{\sigma}))}{\delta\xi^r(\vec{\sigma}')}.\end{aligned} \tag{66}$$

By using Eqs.(40) and (64) of I, we get

$$\begin{aligned}\hat{D}^{(\omega)}_{(a)(b)r}(\tau,\vec{\sigma})\quad&\{\zeta^{(\omega)r}_{(b)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau),{}^3\tilde{M}_{(g)}(\tau,\vec{\sigma}_2)\}=\\ &=-\epsilon_{(a)(d)(b)}\{{}^3\omega_{s(d)}(\tau,\vec{\sigma}),{}^3\tilde{M}_{(g)}(\tau,\vec{\sigma}_2)\}\zeta^{(\omega)s}_{(b)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau),\\ \\ \hat{D}^{(\omega)}_{(a)(b)r}(\tau,\vec{\sigma})\quad&\{\zeta^{(\omega)r}_{(b)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau),{}^3\tilde{\Theta}_u(\tau,\vec{\sigma}_2)\}=\\ &=-\epsilon_{(a)(d)(f)}\{{}^3\omega_{s(d)}(\tau,\vec{\sigma}),{}^3\tilde{\Theta}_u(\tau,\vec{\sigma}_2)\}\zeta^{(\omega)s}_{(f)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau).\end{aligned} \tag{67}$$

Then Eqs.(64) of I, (32) and (40) imply the following transformation properties under rotations and space pseudodiffeomorphisms of the Green function of the SO(3) covariant divergence (which we do not know how to verify explicitly due to the path-ordering contained in it)

$$
\begin{aligned}
\{\zeta^{(\omega)r}_{(a)(b)}(\vec{\sigma},\vec{\sigma}_1;\tau),\quad {}^3\tilde{M}_{(g)}(\tau,\vec{\sigma}_2)\} &= \frac{\partial}{\partial\sigma_2^s}\Big[\zeta^{(\omega)r}_{(a)(e)}(\vec{\sigma},\vec{\sigma}_2;\tau)\epsilon_{(e)(g)(f)}\zeta^{(\omega)s}_{(f)(b)}(\vec{\sigma}_2,\vec{\sigma}_1;\tau)\Big]+\\
&+\zeta^{(\omega)r}_{(a)(e)}(\vec{\sigma},\vec{\sigma}_2)\,{}^3\omega_{s(e)}(\tau,\vec{\sigma}_2)\,\zeta^{(\omega)s}_{(g)(b)}(\vec{\sigma}_2,\vec{\sigma}_1;\tau)-\\
&-\zeta^{(\omega)r}_{(a)(g)}(\vec{\sigma},\vec{\sigma}_2;\tau)\,{}^3\omega_{s(f)}(\tau,\vec{\sigma}_2)\,\zeta^{(\omega)s}_{(f)(b)}(\vec{\sigma}_2,\vec{\sigma}_1;\tau),\\
\\
\{\zeta^{(\omega)r}_{(a)(b)}(\vec{\sigma},\vec{\sigma}_1;\tau),\quad {}^3\tilde{\Theta}_u(\tau,\vec{\sigma}_2)\} &=\\
&=\int d^3\sigma_3\zeta^{(\omega)r}_{(a)(e)}(\vec{\sigma},\vec{\sigma}_3;\tau)\,\epsilon_{(e)(d)(f)}\{{}^3\omega_{s(d)}(\tau,\vec{\sigma}_3),{}^3\tilde{\Theta}_u(\tau,\vec{\sigma}_2)\}\zeta^{(\omega)s}_{(f)(b)}(\vec{\sigma}_3,\vec{\sigma}_1;\tau).
\end{aligned}
\tag{68}
$$

Collecting all previous results, we obtain the following form for the Dirac Hamiltonian

$$
\begin{aligned}
\hat{H}_{(D)ADM} &= \int d^3\sigma\Big[n\hat{\mathcal{H}} - n_{(a)}\,{}^3e_{(a)^r}\,{}^3\tilde{\Theta}_r+\\
&+\lambda_n\,\tilde{\pi}^n+\lambda^{\vec{n}}_{(a)}\tilde{\pi}^{\vec{n}}_{(a)}+\lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)}+\hat{\mu}_{(a)}\,{}^3\tilde{M}_{(a)}\Big](\tau,\vec{\sigma})+\\
&+\zeta_A(\tau)\tilde{\pi}^A(\tau)+\zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau)=\\
&=\int d^3\sigma\Big[n\hat{\mathcal{H}}-n_{(a)}\,{}^3e^r_{(a)}\frac{\partial\xi^s}{\partial\sigma^r}\,\tilde{\pi}^{\vec{\xi}}_s+\lambda_n\tilde{\pi}^n+\\
&+\lambda^{\vec{n}}_{(a)}\tilde{\pi}^{\vec{n}}_{(a)}+\lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)}+(\hat{\mu}_{(b)}B_{(b)(a)}(\alpha_{(e)})+\\
&+n_{(b)}\,{}^3e^r_{(b)}\frac{\partial\alpha_{(a)}}{\partial\sigma^r})\tilde{\pi}^{\vec{\alpha}}_{(a)}\Big](\tau,\vec{\sigma})+\zeta_A(\tau)\tilde{\pi}^A(\tau)+\zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau)=\\
&=\int d^3\sigma\Big[n\hat{\mathcal{H}}-n_{(a)}\,{}^3e^r_{(a)}\frac{\partial\xi^s}{\partial\sigma^r}\,\tilde{\pi}^{\vec{\xi}}_s+\\
&+\lambda_n\tilde{\pi}^n+\lambda^{\vec{n}}_{(a)}\tilde{\pi}^{\vec{n}}_{(a)}+\lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)}+\tilde{\mu}_{(a)}\tilde{\pi}^{\vec{\alpha}}_{(a)}\Big](\tau,\vec{\sigma})+\\
&+\zeta_A(\tau)\tilde{\pi}^A(\tau)+\zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau),
\end{aligned}
\tag{69}
$$

where $\tilde{\mu}_{(a)}$ are new Dirac multipliers.

The phase space action, which usually is incorrectly written without the primary constraints, is

$$
\begin{aligned}
\bar{S} &= \int d\tau d^3\sigma\Big[{}^3\tilde{\pi}^r_{(a)}\partial_\tau\,{}^3e_{(a)r}-n\hat{\mathcal{H}}+n_{(a)}\mathcal{H}_{(a)}-\\
&-\lambda_n\tilde{\pi}^n-\lambda^{\vec{n}}_{(a)}\tilde{\pi}^{\vec{n}}_{(a)}-\lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)}-\mu_{(a)}\,{}^3\tilde{M}_{(a)}\Big](\tau,\vec{\sigma})-\\
&-\zeta_A(\tau)\tilde{\pi}^A(\tau)-\zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau)=\\
&=\int d\tau d^3\sigma\Big[{}^3\tilde{\pi}^r_{(a)}\partial_\tau\,{}^3e_{(a)r}-n\hat{\mathcal{H}}+n_{(a)}\,{}^3e^r_{(a)}\,{}^3\tilde{\Theta}_r-\\
&-\lambda_n\tilde{\pi}^n-\lambda^{\vec{n}}_{(a)}\tilde{\pi}^{\vec{n}}_{(a)}-\lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)}-\hat{\mu}_{(a)}\,{}^3\tilde{M}_{(a)}\Big](\tau,\vec{\sigma})-\\
&-\zeta_A(\tau)\tilde{\pi}^A(\tau)-\zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau)=\\
&=\int d\tau d^3\sigma[{}^3\tilde{\pi}^r_{(a)}\partial_\tau\,{}^3e_{(a)r}-n\hat{\mathcal{H}}+n_{(a)}\,{}^3e^r_{(a)}\frac{\partial\xi^s}{\partial\sigma^r}\tilde{\pi}^{\vec{\xi}}_s-\\
&-\lambda_n\tilde{\pi}^n-\lambda^{\vec{n}}_{(a)}\tilde{\pi}^{\vec{n}}_{(a)}-\lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)}-\tilde{\mu}_{(a)}\,\tilde{\pi}^{\vec{\alpha}}_{(a)}](\tau,\vec{\sigma})-\\
&-\zeta_A(\tau)\tilde{\pi}^A(\tau)-\zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau).
\end{aligned}
\tag{70}
$$

In conclusion the 18-dimensional phase space spanned by ${}^3e_{(a)r}$ and ${}^3\tilde{\pi}^r_{(a)}$ has a global [since $\Sigma_\tau \approx R^3$] canonical basis, in which 12 variables are $\alpha_{(a)}$, $\tilde{\pi}^{\vec{\alpha}}_{(a)} \approx 0$, ξ^r, $\tilde{\pi}^{\vec{\xi}}_r \approx 0$. The remaining 6 variables, hidden in the reduced quantities ${}^3\hat{e}_{(a)r}$, ${}^3\hat{\tilde{\pi}}^r_{(a)}$, are 3 pairs of conjugate Dirac's observables with respect to the gauge transformations in $\bar{\mathcal{G}}_R$, namely they are invariant under Abelianized rotations and space pseudodiffeomorphisms [and, therefore, weakly invariant under the original rotations and space pseudodiffeomorphisms] connected with the identity and obtainable as a succession of infinitesimal gauge transformations. However, since space pseudodiffeomorphisms connect different charts in the atlas of Σ_τ and since $\xi^r(\tau,\vec{\sigma}) = \sigma^r$ means to choose as origin of space pseudodiffeomorphisms an arbitrary chart, the functional form of the Dirac's observables will depend on the chart chosen as origin. This will reflect itself in the freedom of how to parametrize the reduced cotriad ${}^3\hat{e}_{(a)r}(\tau,\vec{\sigma})$ in terms of only 3 independent functions: in each chart 'c' they will be denoted $Q^{(c)}_r(\tau,\vec{\sigma})$ and, if 'c+dc' is a new chart connected to 'c' by an infinitesimal space pseudodiffeomorphism of parameters $\vec{\xi}(\tau,\vec{\sigma})$, then we will have $Q^{(c+dc)}_r(\tau,\vec{\sigma}) = \frac{\partial \xi^s(\tau,\vec{\sigma})}{\partial \sigma^r} Q^{(c)}_s(\tau,\vec{\xi}(\tau,\vec{\sigma}))$.

The real invariants under pseudodiffeomorphisms of a Riemannian 3-manifold $(\Sigma_\tau, {}^3g)$ [for which no explicit basis is known], can be expressed in every chart 'c' as functionals of the 3 independent functions $Q^{(c)}_r(\tau,\vec{\sigma})$. Therefore, these 3 functions give a local (chart-dependent) coordinatization of the space of 3-geometries (superspace or moduli space) $Riem\, \Sigma_\tau / Diff\, \Sigma_\tau$ [32,42].

By using Eqs.(57) and (60) in the Hamiltonian expressions of the 4-tensors of Appendix B of I, we can get the most important 4-tensors on the pseudo-Riemannian 4-manifold $(M^4, {}^4g)$ expressed in terms of $\tilde{\lambda}_A$, $\tilde{\pi}^A \approx 0$, $\tilde{\lambda}_{AB}$, $\tilde{\pi}^{AB} \approx 0$, n, $\tilde{\pi}^n \approx 0$, $n_{(a)}$, $\tilde{\pi}^{\vec{n}}_{(a)} \approx 0$, $\alpha_{(a)}$, $\tilde{\pi}^{\vec{\alpha}}_{(a)} \approx 0$, ξ^r, $\tilde{\pi}^{\vec{\xi}}_r \approx 0$, and of the (non canonically conjugate) Dirac's observables with respect to the action of $\bar{\mathcal{G}}_R$, i.e. ${}^3\hat{e}_{(a)r}$, ${}^3\hat{\tilde{\pi}}^r_{(a)}$. If we could extract from ${}^3\hat{e}_{(a)r}$, ${}^3\hat{\tilde{\pi}}^r_{(a)}$, the Dirac observables with respect to the gauge transformations generated by the superhamiltonian constraint $\hat{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$, then we could express all 4-tensors in terms of these final Dirac observables (the independent Cauchy data of tetrad gravity), of the gauge variables n, $n_{(a)}$, $\alpha_{(a)}$, ξ^r and of the gauge variable associated with $\hat{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$ (see Section V), when all the constraints are satisfied. Therefore, we would get not only a chart-dependent expression of the 4-metrics ${}^4g \in Riem\, M^4$, but also of the 4-geometries in $Riem\, M^4 / Diff\, M^4$.

In the next Section we shall study the simplest charts of the atlas of Σ_τ, namely the 3-orthogonal ones. See Appendix A for more information about special coordinate charts.

IV. THE QUASI-SHANMUGADHASAN CANONICAL TRANSFORMATION IN 3-ORTHOGONAL COORDINATES.

The quasi-Shanmugadhasan canonical transformation [51] ["quasi-" because we are not including the superhamiltonian constraint $\hat{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$] Abelianizing the rotation and pseudodiffeomorphism constraints ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) \approx 0$, ${}^3\tilde{\Theta}_r(\tau,\vec{\sigma}) \approx 0$, will send the canonical basis ${}^3e_{(a)r}(\tau,\vec{\sigma})$, ${}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})$, of $T^*\mathcal{C}_e$ in a new basis whose conjugate pairs are $\Big(\alpha_{(a)}(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma}) \approx 0\Big)$, $\Big(\xi^r(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma}) \approx 0)$ for the gauge sector and $\Big(Q_r(\tau,\vec{\sigma}), \tilde{\Pi}^r(\tau,\vec{\sigma})\Big)$ for the sector of Dirac observables.

Therefore, we must parametrize the Dirac observables ${}^3\hat{e}_{(a)r}(\tau,\vec{\sigma})$ in terms of three functions $Q_r(\tau,\vec{\sigma})$, ${}^3\hat{e}_{(a)r}(\tau,\vec{\sigma}) = {}^3\hat{e}_{(a)r}[Q_s(\tau,\vec{\sigma})]$, and then find how the Dirac observables ${}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma})$ are expressible in terms of $Q_r(\tau,\vec{\sigma})$, $\tilde{\Pi}^r(\tau,\vec{\sigma})$, $\tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma})$, $\tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma})$ [they cannot depend on $\alpha_{(a)}(\tau,\vec{\sigma})$, $\xi_r(\tau,\vec{\sigma})$, because they are Dirac observables]. Since from Eqs.(57) we get

$$\begin{aligned}
{}^3g_{rs}(\tau,\vec{\sigma}) &= {}^3e_{(a)r}(\tau,\vec{\sigma})\, {}^3e_{(a)s}(\tau,\vec{\sigma}) = {}^3\bar{e}_{(a)r}(\tau,\vec{\sigma})\, {}^3\bar{e}_{(a)s}(\tau,\vec{\sigma}) = \\
&= \frac{\partial \xi^u(\tau,\vec{\sigma})}{\partial \sigma^r} \frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^s}\, {}^3\hat{e}_{(a)u}[Q_w(\tau,\vec{\xi}(\tau,\vec{\sigma}))]\, {}^3\hat{e}_{(a)v}[Q_w(\tau,\vec{\xi}(\tau,\vec{\sigma}))] = \\
&= \frac{\partial \xi^u(\tau,\vec{\sigma})}{\partial \sigma^r} \frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^s}\, {}^3\hat{g}_{uv}[Q_w(\tau,\vec{\xi}(\tau,\vec{\sigma}))],
\end{aligned} \tag{71}$$

the new metric ${}^3\hat{g}_{uv}(\tau,\vec{\xi})$ must depend only on the functions $Q_w(\tau,\vec{\xi})$. This shows that the parametrization of ${}^3\hat{e}_{(a)r}(\tau,\vec{\sigma})$ will depend on the chosen system of coordinates, which will be declared the origin $\vec{\xi}(\tau,\vec{\sigma}) = \vec{\sigma}$ of pseudodiffeomorphisms from the given chart. Therefore, each Dirac observable 3-metric ${}^3\hat{g}_{uv}$ is an element of DeWitt superspace [41] for Riemannian 3-manifolds: it defines a 3-geometry on Σ_τ.

The simplest global system of coordinates on $\Sigma_\tau \approx R^3$, where to learn how to construct the quasi-Shanmugadhasan canonical transformation, is the 3-orthogonal one , in which ${}^3\hat{g}_{uv}$ is diagonal. In it we have the parametrization

$$\begin{aligned}
&{}^3\hat{e}_{(a)r}(\tau,\vec{\sigma}) = \delta_{(a)r} Q_r(\tau,\vec{\sigma}) \Rightarrow {}^3\hat{e}^r_{(a)}(\tau,\vec{\sigma}) = \frac{\delta^r_{(a)}}{Q_r(\tau,\vec{\sigma})}, \\
&\quad \Rightarrow \quad {}^3\hat{g}_{rs}(\tau,\vec{\sigma}) = \delta_{rs} Q^2_r(\tau,\vec{\sigma}), \\
&\quad ds^2 = \epsilon\Big([N_{(as)} + n]^2 - [N_{(as)r} + n_r] \sum_u \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^u} \frac{1}{Q^2_u(\vec{\xi})} \frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^u} [N_{(as)s} + n_s]\Big)(d\tau)^2 - \\
&\quad - 2\epsilon [N_{(as)r} + n_r] d\tau d\sigma^r - \epsilon \sum_u \frac{\partial \xi^u}{\partial \sigma^r} Q^2_u(\vec{\xi}) \frac{\partial \xi^u}{\partial \sigma^s} d\sigma^r d\sigma^s = \\
&\quad = \epsilon\Big([N_{(as)} + n]^2 (d\tau)^2 - \delta_{uv}[Q_u \frac{\partial \xi^u}{\partial \sigma^r} d\sigma^r + \frac{1}{Q_u} \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^u}(N_{(as)r} + n_r) d\tau] \\
&\quad [Q_v \frac{\partial \xi^v}{\partial \sigma^s} d\sigma^s + \frac{1}{Q_v} \frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^v}(N_{(as)s} + n_s) d\tau],
\end{aligned} \tag{72}$$

with $Q_r(\tau,\vec{\sigma}) = 1 + h_r(\tau,\vec{\sigma}) > 0$ to avoid singularities. The 3 functions $Q^2_r(\tau,\vec{\sigma})$ give a local parametrization of superspace; the presence of singularities in superspace depends on the boundary conditions for $Q_r(\tau,\vec{\sigma})$, i.e. on the possible existence of stability subgroups (isometries) of the group $\bar{\mathcal{G}}$ of gauge transformations, which we assume to be absent if a suitable weighted Sobolev space is chosen for cotriads.

Let us remark that if we change the parametrization of ${}^3\hat{e}_{(a)r}$, giving it as a different function of 3 $\check{Q}_r$, this amounts to a canonical transformation $Q_r, \tilde{\Pi}^r \mapsto \check{Q}_r, \check{\tilde{\Pi}}^r$ with $\delta_{(a)r} Q_r = {}^3\hat{e}_{(a)r}[\check{Q}_s]$ together with a redefinition of the origin of space pseudodiffeomorphism [the new global chart is the new origin defined as $\vec{\xi}^{\,\prime}(\tau,\vec{\sigma}) = \vec{\sigma}$ with $\vec{\xi}^{\,\prime}$ that functional of $\vec{\xi}$ dictated by the pseudodiffeomorphism connecting the two global charts; however, $\vec{\xi}^{\,\prime}$ can be renamed $\vec{\xi}$ being a canonical variable of our basis]. In the quasi-Shanmugadhasan canonical transformation we will study in this Section, this will be reflected in the change of the expression giving ${}^3\tilde{\pi}^r_{(a)}$ in terms of the new variables.

The choice of the parametrization of ${}^3\hat{e}_{(a)r}$ is equivalent to the coordinate conditions of Refs. [52,4]. See Eqs.(A5) of Appendix A for a parametrization of the cotriads ${}^3\hat{e}_{(a)r}$ corresponding to normal coordinates around the point $\{\tau, \vec{\sigma} = 0\} \in \Sigma_\tau$.

Since the rotation constraints ${}^3\tilde{M}_{(a)} = \epsilon_{(a)(b)(c)}\, {}^3e_{(b)r}\, {}^3\tilde{\pi}^r_{(c)} = \frac{1}{2}\epsilon_{(a)(b)(c)}\, {}^3\tilde{M}_{(b)(c)}$ may be written as

$$
{}^3\tilde{M}_{(a)(b)} = {}^3e_{(a)r}\, {}^3\tilde{\pi}^r_{(b)} - {}^3e_{(b)r}\, {}^3\tilde{\pi}^r_{(a)} = \epsilon_{(a)(b)(c)}\, {}^3\tilde{M}_{(c)} = \epsilon_{(a)(b)(c)}\, \tilde{\pi}^{\vec{\alpha}}_{(d)} B_{(d)(c)}(\alpha_{(e)})
$$

due to Eqs.(39), we may extract the dependence of ${}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})$ from $\tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma})$

$$
\begin{aligned}
{}^3\tilde{\pi}^r_{(a)} &= {}^3e^r_{(b)}\, {}^3e_{(b)s}\, {}^3\tilde{\pi}^s_{(a)} = \\
&= \frac{1}{2}\, {}^3e^r_{(b)} \Big({}^3e_{(b)s}\, {}^3\tilde{\pi}^s_{(a)} + {}^3e_{(a)s}\, {}^3\tilde{\pi}^s_{(b)} \Big) + \frac{1}{2}\, {}^3e^r_{(b)} \Big({}^3e_{(b)s}\, {}^3\tilde{\pi}^s_{(a)} - {}^3e_{(a)s}\, {}^3\tilde{\pi}^s_{(b)} \Big) = \\
&= \frac{1}{2}\, {}^3e^r_{(b)} \Big({}^3e_{(b)s}\, {}^3\tilde{\pi}^s_{(a)} + {}^3e_{(a)s}\, {}^3\tilde{\pi}^s_{(b)} \Big) - \\
&- \frac{1}{2}\, {}^3e^r_{(b)}\, \epsilon_{(a)(b)(c)} \tilde{\pi}^{\vec{\alpha}}_{(d)} B_{(d)(c)}(\alpha_{(e)}) \stackrel{def}{=}, \\
&\stackrel{def}{=} \frac{1}{2}\, {}^3e^r_{(b)} \Big[Z_{(a)(b)} - \epsilon_{(a)(b)(c)} \tilde{\pi}^{\vec{\alpha}}_{(d)} B_{(d)(c)}(\alpha_{(e)}) \Big]
\end{aligned}
$$

$$
Z_{(a)(b)} = Z_{(b)(a)} = {}^3e_{(a)s}\, {}^3\tilde{\pi}^s_{(b)} + {}^3e_{(b)s}\, {}^3\tilde{\pi}^s_{(a)} = Z_{(a)(b)}[\alpha_{(e)}, \xi^r, \tilde{\pi}^{\vec{\xi}}_r, Q_r, \tilde{\Pi}^r]. \tag{73}
$$

To extract the dependence of ${}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})$ on $\tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma})$, let us recall Eqs.(62) of I, (39) and (51)

$$
\begin{aligned}
\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) &= \hat{D}^{(\omega)}_{(a)(b)r}(\tau,\vec{\sigma})\, {}^3\tilde{\pi}^r_{(b)}(\tau,\vec{\sigma}) = \\
&= -{}^3e^r_{(a)}(\tau,\vec{\sigma}) \Big[{}^3\tilde{\Theta}_r + {}^3\omega_{r(b)}\, {}^3\tilde{M}_{(b)} \Big](\tau,\vec{\sigma}) = \\
&= -{}^3e^r_{(a)}(\tau,\vec{\sigma}) \Big[\frac{\partial \xi^s}{\partial \sigma^r} \tilde{\pi}^{\vec{\xi}}_s + (B_{(b)(c)}(\alpha_{(e)})\, {}^3\omega_{r(c)} - \frac{\partial \alpha_{(b)}}{\partial \sigma^r})\, \tilde{\pi}^{\vec{\alpha}}_{(b)} \Big](\tau,\vec{\sigma}),
\end{aligned} \tag{74}
$$

and Eqs.(39)

$$
{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) = {}^3\tilde{\pi}^{(T)r}_{(a)}(\tau,\vec{\sigma}) - \int d^3\sigma_1\, \zeta^{(\omega)r}_{(a)(b)}(\vec{\sigma},\vec{\sigma}_1;\tau)\, \hat{\mathcal{H}}_{(b)}(\tau,\vec{\sigma}_1). \tag{75}
$$

Then we have

$$
\begin{aligned}
&\Big[Z_{(a)(b)}-\epsilon_{(a)(b)(c)}\tilde{\pi}^{\vec{\alpha}}_{(d)}B_{(d)(c)}(\alpha_{(e)})\Big](\tau,\vec{\sigma})=2\Big[{}^3e_{(b)r}\,{}^3\tilde{\pi}^r_{(a)}\Big](\tau,\vec{\sigma})=\\
&\qquad=2\Big[{}^3e_{(b)r}\,{}^3\tilde{\pi}^{(T)r}_{(a)}\Big](\tau,\vec{\sigma})-2{}^3e_{(b)r}(\tau,\vec{\sigma})\int d^3\sigma_1\,\zeta^{(\omega)r}_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\hat{\mathcal{H}}_{(c)}(\tau,\vec{\sigma}_1)=\\
&\qquad=S_{(a)(b)}(\tau,\vec{\sigma})-\\
&\qquad-\int d^3\sigma_1\Big[{}^3e_{(b)r}(\tau,\vec{\sigma})\,\zeta^{(\omega)r}_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)+{}^3e_{(a)r}(\tau,\vec{\sigma})\,\zeta^{(\omega)r}_{(b)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big]\hat{\mathcal{H}}_{(c)}(\tau,\vec{\sigma}_1)+\\
&\qquad+\Big[{}^3e_{(b)r}\,{}^3\tilde{\pi}^{(T)r}_{(a)}-{}^3e_{(a)r}\,{}^3\tilde{\pi}^{(T)r}_{(b)}\Big](\tau,\vec{\sigma})-\\
&\qquad-\int d^3\sigma_1\Big[{}^3e_{(b)r}(\tau,\vec{\sigma})\,\zeta^{(\omega)r}_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)-{}^3e_{(a)r}(\tau,\vec{\sigma})\,\zeta^{(\omega)r}_{(b)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big]\hat{\mathcal{H}}_{(c)}(\tau,\vec{\sigma}_1),
\end{aligned}
\tag{76}
$$

with

$$
S_{(a)(b)}(\tau,\vec{\sigma})=S_{(b)(a)}(\tau,\vec{\sigma})=\Big[{}^3e_{(a)r}\,{}^3\tilde{\pi}^{(T)r}_{(b)}+{}^3e_{(b)r}\,{}^3\tilde{\pi}^{(T)r}_{(a)}\Big](\tau,\vec{\sigma}).
\tag{77}
$$

By equating the terms symmetric and antisymmetric in $(a)\Leftrightarrow(b)$, we get

$$
\begin{aligned}
&Z_{(a)(b)}(\tau,\vec{\sigma})=S_{(a)(b)}(\tau,\vec{\sigma})-\\
&-\int d^3\sigma_1\Big[{}^3e_{(b)r}(\tau,\vec{\sigma})\,\zeta^{(\omega)r}_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)+{}^3e_{(a)r}(\tau,\vec{\sigma})\,\zeta^{(\omega)r}_{(b)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big]\hat{\mathcal{H}}_{(c)}(\tau,\vec{\sigma}_1),\\
&\\
&\Big[{}^3e_{(a)r}\,{}^3\tilde{\pi}^{(T)r}_{(b)}-{}^3e_{(b)r}\,{}^3\tilde{\pi}^{(T)r}_{(a)}\Big](\tau,\vec{\sigma})=\epsilon_{(a)(b)(c)}\big[\tilde{\pi}^{\vec{\alpha}}_{(d)}B_{(d)(c)}(\alpha_{(e)})\Big](\tau,\vec{\sigma})+\\
&+\int d^3\sigma_1\Big[{}^3e_{(b)r}(\tau,\vec{\sigma})\,\zeta^{(\omega)r}_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)-{}^3e_{(a)r}(\tau,\vec{\sigma})\,\zeta^{(\omega)r}_{(b)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big]\hat{\mathcal{H}}_{(c)}(\tau,\vec{\sigma}_1),
\end{aligned}
\tag{78}
$$

so that we obtain the following dependence of ${}^3\tilde{\pi}^r_{(a)}$ on $\tilde{\pi}^{\vec{\alpha}}_{(a)}$ and $\tilde{\pi}^{\vec{\xi}}_r$

$$
\begin{aligned}
{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})&=\frac{1}{2}{}^3e^r_{(b)}(\tau,\vec{\sigma})\Big(S_{(a)(b)}(\tau,\vec{\sigma})-\epsilon_{(a)(b)(c)}\tilde{\pi}^{\vec{\alpha}}_{(d)}(\tau,\vec{\sigma})B_{(d)(c)}(\alpha_{(e)}(\tau,\vec{\sigma}))-\\
&-\int d^3\sigma_1\Big[{}^3e_{(b)u}(\tau,\vec{\sigma})\,\zeta^{(\omega)u}_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)+{}^3e_{(a)u}(\tau,\vec{\sigma})\,\zeta^{(\omega)u}_{(b)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big]\cdot\\
&\cdot\;{}^3e^w_{(c)}(\tau,\vec{\sigma}_1)\Big[\frac{\partial\xi^s}{\partial\sigma^w_1}\tilde{\pi}^{\vec{\xi}}_s+\big(B_{(d)(f)}(\alpha_{(e)})\,{}^3\omega_{w(f)}-\frac{\partial\alpha_{(d)}}{\partial\sigma^w_1}\big)\tilde{\pi}^{\vec{\alpha}}_{(d)}\Big](\tau,\vec{\sigma}_1)\Big).
\end{aligned}
\tag{79}
$$

Therefore, all the dependence of ${}^3\tilde{\pi}^r_{(a)}$ on $\tilde{\Pi}^r$ is hidden in $S_{(a)(b)}$. To find it, let us impose the canonicity of the transformation

${}^3e_{(a)r},{}^3\tilde{\pi}^r_{(a)}\mapsto\alpha_{(a)},\tilde{\pi}^{\vec{\alpha}}_{(a)},\xi^r,\tilde{\pi}^{\vec{\xi}}_r,Q_r,\tilde{\Pi}^r$

by taking into account that

$\{\alpha_{(a)}(\tau,\vec{\sigma}),\tilde{\pi}^{\vec{\alpha}}_{(b)}(\tau,\vec{\sigma}^{'})\}=\delta_{(a)(b)}\delta^3(\vec{\sigma},\vec{\sigma}^{'})$,
$\{\xi^r(\tau,\vec{\sigma}),\tilde{\pi}^{\vec{\xi}}_s(\tau,\vec{\sigma}^{'})\}=\{Q_s(\tau,\vec{\sigma}),\tilde{\Pi}^r(\tau,\vec{\sigma}^{'})\}=\delta^r_s\delta^3(\vec{\sigma},\vec{\sigma}^{'})$
and that $\{{}^3e_{(a)r}(\tau,\vec{\sigma}),\alpha_{(b)}(\tau,\vec{\sigma}^{'})\}=\{{}^3e_{(a)r}(\tau,\vec{\sigma}),\xi_s(\tau,\vec{\sigma}^{'})\}=\{{}^3e_{(a)r}(\tau,\vec{\sigma}),Q_s(\tau,\vec{\sigma}^{'})\}=0$

$$
\begin{aligned}
\delta^s_r\delta_{(a)(b)}\delta^3(\vec{\sigma},{\vec{\sigma}}^{'}) &= \{ {}^3e_{(a)r}(\tau ,\vec \sigma), {}^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})\} =\\
&= \int d^3\sigma_1\Big[\{ {}^3e_{(a)r}(\tau ,\vec \sigma), {\tilde \pi}^{\vec \alpha}_{(c)}(\tau ,{\vec \sigma}_1)\} \{ \alpha_{(c)}(\tau ,{\vec \sigma}_1), {}^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})\} +\\
&+ \{ {}^3e_{(a)r}(\tau ,\vec \sigma), {\tilde \pi}^{\vec \xi}_u(\tau ,{\vec \sigma}_1)\} \{ \xi^u(\tau ,{\vec \sigma}_1), {}^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})\} +\\
&+ \{ {}^3e_{(a)r}(\tau ,\vec \sigma), {\tilde \Pi}^u(\tau ,{\vec \sigma}_1)\} \{ Q_u(\tau ,{\vec \sigma}_1), {}^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})\} \Big] =\\
&= \int d^3\sigma_1\Big[\frac{\tilde \delta^3 e_{(a)r}(\tau ,\vec \sigma)}{\delta \alpha_{(c)}(\tau ,{\vec \sigma}_1)}\frac{\delta^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})}{\delta {\tilde \pi}^{\vec \alpha}_{(c)}(\tau ,{\vec \sigma}_1)} + \frac{\delta^3e_{(a)r}(\tau ,\vec \sigma)}{\delta \xi^u(\tau ,{\vec \sigma}_1)}\frac{\delta^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})}{\delta {\tilde \pi}^{\vec \xi}_u(\tau ,{\vec \sigma}_1)} +\\
&+ \frac{\delta^3e_{(a)r}(\tau ,\vec \sigma)}{\delta Q_u(\tau ,{\vec \sigma}_1)}\frac{\delta^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})}{\delta {\tilde \Pi}^u(\tau ,{\vec \sigma}_1)}\Big] . \qquad (80)
\end{aligned}
$$

[we could replace $\alpha_{(a)}(\tau ,\vec \sigma)$ with $\alpha_{(a)}(\tau ,\vec \xi (\tau ,\vec \sigma))$, since the angles are scalar fields under pseudodiffeomorphims].

Since ${}^3e_{(a)r}(\tau ,\vec \sigma) = {}^3R_{(a)(b)}(\alpha_{(e)}(\tau ,\vec \sigma)) \frac{\partial \xi^u}{\partial \sigma^r}\delta_{(b)u} Q_u(\tau ,\vec \xi (\tau ,\vec \sigma))$, by using Eqs.(30) and $H_{(a)}(\alpha_{(e)}) = {\hat R}^{(d)} A_{(d)(a)}(\alpha_{(e)})$ we get

$$
\begin{aligned}
\frac{\tilde \delta^3e_{(a)r}(\tau ,\vec \sigma)}{\delta \alpha_{(c)}(\tau ,{\vec \sigma}_1)} &= \delta^3(\vec \sigma ,{\vec \sigma}_1) \Big[H_{(c)}(\alpha_{(e)}(\tau ,\vec \sigma)) {}^3R(\alpha_{(e)}(\tau ,\vec \sigma))\Big]_{(a)(b)}\\
&\quad \sum_u \frac{\partial \xi^u(\tau ,\vec \sigma)}{\partial \sigma^r}\delta_{(b)u} Q_u(\tau ,\vec \xi (\tau ,\vec \sigma)) =\\
&= \delta^3(\vec \sigma ,{\vec \sigma}_1) \epsilon_{(a)(n)(d)} A_{(d)(c)}(\alpha_{(e)}(\tau ,\vec \sigma)) {}^3R_{(n)(m)}(\alpha_{(e)}(\tau ,\vec \sigma)) \cdot\\
&\cdot \sum_u \frac{\partial \xi^u(\tau ,\vec \sigma)}{\partial \sigma^r}\delta_{(m)u} Q_u(\tau ,\vec \xi (\tau ,\vec \sigma)),\\
\frac{\delta^3e_{(a)r}(\tau ,\vec \sigma)}{\delta \xi^u(\tau ,{\vec \sigma}_1)} &= {}^3R_{(a)(n)}(\alpha_{(e)}(\tau ,\vec \sigma)) \sum_v \delta_{(n)v} \Big[\frac{\partial \xi^v(\tau ,\vec \sigma)}{\partial \sigma^r} \frac{\partial Q_v(\tau ,\vec \xi)}{\partial \xi^u}{|}_{\vec \xi = \vec \xi (\tau ,\vec \sigma)} \delta^3(\vec \sigma ,{\vec \sigma}_1) +\\
&+ \delta^v_u Q_v(\tau ,\vec \xi (\tau ,\vec \sigma)) \frac{\partial \delta^3(\vec \sigma ,{\vec \sigma}_1)}{\partial \sigma^r}\Big] ,\\
\frac{\delta^3e_{(a)r}(\tau ,\vec \sigma)}{\delta Q_u(\tau ,{\vec \sigma}_1)} &= {}^3R_{(a)(n)}(\alpha_{(e)}(\tau ,\vec \sigma)) \sum_v \frac{\partial \xi^v(\tau ,\vec \sigma)}{\partial \sigma^r} \delta_{(n)v} \delta^u_v \delta^3(\vec \xi (\tau ,\vec \sigma),{\vec \sigma}_1). \qquad (81)
\end{aligned}
$$

Then, Eqs.(79) give

$$
\begin{aligned}
\frac{\delta^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})}{\delta {\tilde \pi}^{\vec \alpha}_{(c)}(\tau ,{\vec \sigma}_1)} &= \frac{1}{2} {}^3e^s_{(h)}(\tau ,{\vec \sigma}^{'}) \Big(-\epsilon_{(b)(h)(k)} B_{(c)(k)}(\alpha_{(e)}(\tau ,{\vec \sigma}^{'})) \delta^3({\vec \sigma}_1,{\vec \sigma}^{'}) -\\
&- \Big[{}^3e_{(b)t}(\tau ,{\vec \sigma}^{'})\, \zeta^{(\omega)t}_{(h)(k)}({\vec \sigma}^{'},{\vec \sigma}_1;\tau) + {}^3e_{(h)t}(\tau ,{\vec \sigma}^{'})\, \zeta^{(\omega)t}_{(b)(k)}({\vec \sigma}^{'},{\vec \sigma}_1;\tau)\Big] \cdot\\
&\cdot\ {}^3e^w_{(k)}(\tau ,{\vec \sigma}_1) \Big[B_{(c)(f)}(\alpha_{(e)})\, {}^3\omega_{w(f)} - \frac{\partial \alpha_{(c)}}{\partial \sigma_1^w}\Big] (\tau ,{\vec \sigma}_1)\Big) ,\\
\frac{\delta^3{\tilde \pi}^s_{(b)}(\tau ,{\vec \sigma}^{'})}{\delta {\tilde \pi}^{\vec \xi}_u(\tau ,{\vec \sigma}_1)} &= -\frac{1}{2} {}^3e^s_{(h)}(\tau ,{\vec \sigma}^{'}) \Big[{}^3e_{(b)t}(\tau ,{\vec \sigma}^{'})\, \zeta^{(\omega)t}_{(h)(k)}({\vec \sigma}^{'},{\vec \sigma}_1;\tau) +\\
&+ {}^3e_{(h)t}(\tau ,{\vec \sigma}^{'})\, \zeta^{(\omega)t}_{(b)(k)}({\vec \sigma}^{'},{\vec \sigma}_1;\tau)\Big] {}^3e^w_{(k)}(\tau ,{\vec \sigma}_1) \frac{\partial \xi^u(\tau ,{\vec \sigma}_1)}{\partial \sigma_1^w},
\end{aligned}
$$

$$\frac{\delta^3 \tilde{\pi}^s_{(b)}(\tau,\vec{\sigma}^{'})}{\delta \tilde{\Pi}^u(\tau,\vec{\sigma}_1)} = \frac{1}{2}\, {}^3e^s_{(h)}(\tau,\vec{\sigma}^{'}) \frac{\delta S_{(b)(h)}(\tau,\vec{\sigma}^{'})}{\delta \tilde{\Pi}^u(\tau,\vec{\sigma}_1)}. \tag{82}$$

so that Eqs.(80) become

$$\begin{aligned}
\delta^s_r \delta_{(a)(b)} \quad & \delta^3(\vec{\sigma},\vec{\sigma}^{'}) = \\
& = -\frac{1}{2}\epsilon_{(a)(n)(d)}\, {}^3R_{(n)(m)}(\alpha_{(e)}(\tau,\vec{\sigma})) \sum_v \frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^r} \delta_{(m)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma})) \cdot \\
& \cdot\ {}^3e^s_{(h)}(\tau,\vec{\sigma}^{'}) \Big(\epsilon_{(b)(h)(d)}\delta^3(\vec{\sigma},\vec{\sigma}^{'}) + {}^3e^w_{(k)}(\tau,\vec{\sigma}) \Big[{}^3\omega_{w(d)}(\tau,\vec{\sigma}) - A_{(d)(c)}(\alpha_{(e)}(\tau,\vec{\sigma})) \cdot \\
& \cdot\ \frac{\partial \alpha_{(c)}(\tau,\vec{\sigma})}{\partial \sigma^w} \Big] T_{(b)(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) \Big) - \\
& - \frac{1}{2}\, {}^3R_{(a)(n)}(\alpha_{(e)}(\tau,\vec{\sigma})) \sum_v \delta_{(n)v} \Big(\frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^r}\, {}^3e^w_{(k)}(\tau,\vec{\sigma}) \cdot \\
& \cdot\ \frac{\partial Q_v(\tau,\vec{\xi})}{\partial \sigma^w} {}^3e^s_{(h)}(\tau,\vec{\sigma}^{'}) T_{(b)(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) + \\
& + Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))\, {}^3e^s_{(h)}(\tau,\vec{\sigma}^{'}) \\
& \cdot\ \frac{\partial}{\partial \sigma^r} \Big[{}^3e^w_{(k)}(\tau,\vec{\sigma}) \frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^w} T_{(b)(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) \Big] \Big) + \\
& + \frac{1}{2}\, {}^3R_{(a)(n)}(\alpha_{(e)}(\tau,\vec{\sigma})) \sum_v \frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^r} \delta_{(n)v}\, {}^3e^s_{(h)}(\tau,\vec{\sigma}^{'}) \frac{\delta S_{(h)(b)}(\tau,\vec{\sigma}^{'})}{\delta \tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma}))},
\end{aligned} \tag{83}$$

where we introduced the notation

$$\begin{aligned}
T_{(b)(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) &= {}^3e_{(b)t}(\tau,\vec{\sigma}^{'})\, \zeta^{(\omega)t}_{(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) + {}^3e_{(h)t}(\tau,\vec{\sigma}^{'})\, \zeta^{(\omega)t}_{(b)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) = \\
&= \sum_r Q_r(\tau,\vec{\sigma}^{'}) \Big[\delta_{(b)r} \zeta^{(\omega)r}_{(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) + \delta_{(h)r} \zeta^{(\omega)r}_{(b)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) \Big] = \\
&= \sum_r Q_r(\tau,\vec{\sigma}) d^r_{\gamma_{P'P}}(\vec{\sigma}^{'},\vec{\sigma}) \\
&\quad \Big[\delta_{(b)r} \Big(P_{\gamma_{P'P}}\, e^{\int_{\vec{\sigma}}^{\vec{\sigma}^{'}} d\sigma_1^s\, \hat{R}^{(c)}\, {}^3\omega_{s(c)}(\tau,\vec{\sigma}_1)} \Big)_{(h)(k)} + \\
&+ \delta_{(h)r} \Big(P_{\gamma_{P'P}}\, e^{\int_{\vec{\sigma}}^{\vec{\sigma}^{'}} d\sigma_1^s\, \hat{R}^{(c)}\, {}^3\omega_{s(c)}(\tau,\vec{\sigma}_1)} \Big)_{(b)(k)} \Big].
\end{aligned} \tag{84}$$

By multiplying this equation by ${}^3R^{-1}_{(g)(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) = {}^3R^T_{(g)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))$ and then by sending $(g) \mapsto (a)$, we get

$$\begin{aligned}
\sum_v \frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^r} \delta_{(a)v} \quad & {}^3e^s_{(h)}(\tau,\vec{\sigma}^{'}) \frac{\delta S_{(h)(b)}(\tau,\vec{\sigma}^{'})}{\delta \tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma}))} = \\
& = 2\delta^s_r\, {}^3R_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))\delta^3(\vec{\sigma},\vec{\sigma}^{'}) + \\
& + {}^3R_{(f)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}))\epsilon_{(f)(n)(d)}\, {}^3R_{(n)(m)}(\alpha_{(e)}(\tau,\vec{\sigma})) \sum_v \frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^r} \cdot \\
& \cdot\ \delta_{(m)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))\, {}^3e^s_{(h)}(\tau,\vec{\sigma}^{'}) \Big[\epsilon_{(b)(h)(d)}\delta^3(\vec{\sigma},\vec{\sigma}^{'}) +
\end{aligned}$$

$$
\begin{aligned}
&+ {}^3e^w_{(k)}(\tau,\vec{\sigma})({}^3\omega_{w(d)}(\tau,\vec{\sigma}) - A_{(d)(c)}(\alpha_{(e)}(\tau,\vec{\sigma}))\frac{\partial\alpha_{(c)}(\tau,\vec{\sigma})}{\partial\sigma^w})\\
&\cdot\ T_{(b)(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau)\Big] +\\
&+\sum_v \delta_{(a)v}\frac{\partial\xi^v(\tau,\vec{\sigma})}{\partial\sigma^r}\frac{\partial Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))}{\partial\sigma^w}\,{}^3e^w_{(k)}(\tau,\vec{\sigma})\,{}^3e^s_{(h)}(\tau,\vec{\sigma}^{'})\\
&\cdot\ T_{(b)(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) +\\
&+\sum_v \delta_{(a)v}Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))\,{}^3e^s_{(h)}(\tau,\vec{\sigma}^{'})\\
&\cdot\ \frac{\partial}{\partial\sigma^r}\Big[{}^3e^w_{(k)}(\tau,\vec{\sigma})\frac{\partial\xi^v(\tau,\vec{\sigma})}{\partial\sigma^w}T_{(b)(h)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau)\Big].
\end{aligned}
\tag{85}
$$

From Eqs.(30) and (31) and from Eq.(5.32) of the second paper in Ref. [2], we have

$$
{}^3R_{(f)(a)}\epsilon_{(f)(n)(d)}\,{}^3R_{(n)(m)} = [{}^3R^{-1}\,\hat{R}^{(d)}\,{}^3R]_{(a)(m)} = (\hat{R}^{(n)})_{(a)(m)}\,{}^3R_{(n)(d)} = \epsilon_{(a)(m)(n)}\,{}^3R_{(n)(d)}
$$

and, then, $\epsilon_{(a)(m)(n)}\,{}^3R_{(n)(d)}\epsilon_{(b)(g)(d)} = -[\hat{R}^{(a)}\,{}^3R\,\hat{R}^{(b)}]_{(m)(g)}$. Then, by multiplying the previous equation by ${}^3e_{(g)s}(\tau,\vec{\sigma}^{'})$ we obtain [also using Eq.(33)]

$$
\begin{aligned}
\sum_v \frac{\partial\xi^v(\tau,\vec{\sigma})}{\partial\sigma^r}\quad & \delta_{(a)v}\frac{\delta S_{(g)(b)}(\tau,\vec{\sigma}^{'})}{\delta\tilde{\Pi}^v(\tau,\vec{\sigma}))} =\\
&= \Big(2\,{}^3e_{(g)r}(\tau,\vec{\sigma})\,{}^3R_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) - \Big[\hat{R}^{(a)}\,{}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\,\hat{R}^{(b)}\Big]_{(m)(g)}\cdot\\
&\cdot\ \sum_v \frac{\partial\xi^v(\tau,\vec{\sigma})}{\partial\sigma^r}\delta_{(m)v}Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))\Big)\,\delta^3(\vec{\sigma},\vec{\sigma}^{'}) +\\
&+\sum_v \frac{\partial\xi^v(\tau,\vec{\sigma})}{\partial\sigma^r}{}^3e^w_{(k)}(\tau,\vec{\sigma})\Big(\Big[\hat{R}^{(a)}\,{}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\Big]_{(m)(d)}\delta_{(m)v}Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))\cdot\\
&\cdot\ {}^3\omega^{(T)}_{w(d)}(\tau,\vec{\sigma},\alpha_{(e)}(\tau,\vec{\sigma})) +\\
&+\sum_v \delta_{(a)v}\,\frac{\partial Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))}{\partial\sigma^w}\Big)T_{(b)(g)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) +\\
&+\sum_v \delta_{(a)v}Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))\frac{\partial}{\partial\sigma^r}\Big[{}^3e^w_{(k)}(\tau,\vec{\sigma})\frac{\partial\xi^v(\tau,\vec{\sigma})}{\partial\sigma^w}T_{(b)(g)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau)\Big],
\end{aligned}
\tag{86}
$$

and by multiplication by $\frac{\partial\sigma^r(\vec{\xi})}{\partial\xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}$ we have [there is no sum over u]

$$
\begin{aligned}
&\frac{\delta S_{(g)(b)}(\tau,\vec{\sigma}^{'})}{\delta[\delta_{(a)u}\,\tilde{\Pi}^u(\tau,\vec{\xi}(\tau,\vec{\sigma}))]} = \delta_{(a)u}\frac{\delta S_{(g)(b)}(\tau,\vec{\sigma}^{'})}{\delta\tilde{\Pi}^u(\tau,\vec{\xi}(\tau,\vec{\sigma}))} =\\
&= \Big(2\,{}^3e_{(g)r}(\tau,\vec{\sigma})\frac{\partial\sigma^r(\vec{\xi})}{\partial\xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\,{}^3R_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma})) -\\
&- \Big[\hat{R}^{(a)}\,{}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\,\hat{R}^{(b)}\Big]_{(m)(g)}\delta_{(m)u}Q_u(\tau,\vec{\xi}(\tau,\vec{\sigma}))\Big)\delta^3(\vec{\sigma},\vec{\sigma}^{'}) +
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(\Big[\hat{R}^{(a)}\, {}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\Big]_{(m)(d)} \delta_{(m)u} Q_u(\tau, \vec{\xi}(\tau,\vec{\sigma}))\, {}^3\omega^{(T)}_{w(d)}(\tau,\vec{\sigma},\alpha_{(e)}(\tau,\vec{\sigma})) + \\
&+ \delta_{(a)u} \frac{\partial Q_u(\tau,\vec{\xi}(\tau,\vec{\sigma}))}{\partial \sigma^w} \Big) {}^3e^w_{(k)}(\tau,\vec{\sigma})\, T_{(b)(g)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) + \\
&+ \sum_v \delta_{(a)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma})) \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^u}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})} \frac{\partial}{\partial \sigma^r} \Big[{}^3e^w_{(k)}(\tau,\vec{\sigma}) \frac{\partial \xi^v(\tau,\vec{\sigma})}{\partial \sigma^w} T_{(b)(g)(k)}(\vec{\sigma}^{'},\vec{\sigma};\tau) \Big] = \\
&= \sum_v \Big(2\, {}^3e_{(g)r}(\tau,\vec{\sigma}) \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\, {}^3R_{(b)(c)}(\alpha_{(e)}(\tau,\vec{\sigma})) - \\
&- \Big[\hat{R}^{(c)}\, {}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\, \hat{R}^{(b)}\Big]_{(m)(g)} \delta_{(m)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma})) \Big)\ \delta_{(c)(a)} \delta_{(a)u} \delta^v_u \delta^3(\vec{\sigma},\vec{\sigma}^{'}) + \\
&+ \int d^3\sigma_1\, \delta_{(a)(c)} \sum_v \delta^v_u \delta^3(\vec{\sigma},\vec{\sigma}_1) \delta_{(a)u} \cdot \\
&\cdot\ \Big({}^3e^w_{(k)}(\tau,\vec{\sigma}_1)\, T_{(b)(g)(k)}(\vec{\sigma}^{'},\vec{\sigma}_1;\tau) \cdot \\
&\cdot\ \Big[[\hat{R}^{(c)}\, {}^3R(\alpha_{(e)}(\tau,\vec{\sigma}_1))]_{(m)(d)} \delta_{(m)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))\, {}^3\omega^{(T)}_{w(d)}(\tau,\vec{\sigma}_1,\alpha_{(e)}(\tau,\vec{\sigma}_1)) + \\
&+ \delta_{(c)v} \frac{\partial Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))}{\partial \sigma_1^w} \Big] + \\
&+ \sum_t \delta_{(c)t} Q_t(\tau,\vec{\xi}(\tau,\vec{\sigma}_1)) \frac{\partial \sigma_1^r(\vec{\xi})}{\partial \xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma}_1)} \frac{\partial}{\partial \sigma_1^r} \Big[{}^3e^w_{(k)}(\tau,\vec{\sigma}_1) \frac{\partial \xi^t(\tau,\vec{\sigma}_1)}{\partial \sigma_1^w} T_{(b)(g)(k)}(\vec{\sigma}^{'},\vec{\sigma}_1;\tau) \Big] \Big).
\end{aligned}
\tag{87}
$$

This is the final equation for $S_{(a)(b)}$ in terms of $\tilde{\Pi}^u$. Since we have [no sum over u,v]

$$
\begin{aligned}
\frac{\delta\, [\delta_{(c)v} \tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma}^{'}))]}{\delta\, [\delta_{(a)u} \tilde{\Pi}^u(\tau,\vec{\xi}(\tau,\vec{\sigma}))]} &= \delta_{(a)u} \delta_{(c)v} \delta^v_u \delta^3(\vec{\xi}(\tau,\vec{\sigma}),\vec{\xi}(\tau,\vec{\sigma}^{'})) = \\
&= \delta_{(a)(c)} \delta_{(a)u} \delta^v_u \frac{\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{|\frac{\partial \xi}{\partial \sigma}(\tau,\vec{\sigma})|},
\end{aligned}
\tag{88}
$$

the final solution for $S_{(a)(b)}$ is

$$
\begin{aligned}
&S_{(a)(b)}(\tau,\vec{\sigma}) = \\
&= |\, \frac{\partial \xi}{\partial \sigma}(\tau,\vec{\sigma})\,| \cdot \Big(\sum_v \Big[2\, {}^3e_{(a)r}(\tau,\vec{\sigma}) \frac{\partial \sigma^r(\vec{\xi})}{\partial \xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\, {}^3R_{(b)(c)}(\alpha_{(e)}(\tau,\vec{\sigma})) - \\
&- [\hat{R}^{(c)}\, {}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\, \hat{R}^{(b)}]_{(m)(a)} \delta_{(m)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma})) \Big]\, \delta_{(c)v} \tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma})) + \\
&+ \int d^3\sigma_1\, \sum_v \delta_{(c)v} \tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))\, \frac{|\frac{\partial \xi}{\partial \sigma_1}(\tau,\vec{\sigma}_1)|}{|\frac{\partial \xi}{\partial \sigma}(\tau,\vec{\sigma})|} \cdot \\
&\cdot\ \Big({}^3e^w_{(k)}(\tau,\vec{\sigma}_1)\, T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) \cdot \\
&\cdot\ \Big[[\hat{R}^{(c)}\, {}^3R(\alpha_{(e)}(\tau,\vec{\sigma}_1))]_{(m)(d)} \delta_{(m)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))\, {}^3\omega^{(T)}_{w(d)}(\tau,\vec{\sigma}_1,\alpha_{(e)}(\tau,\vec{\sigma}_1)) + \\
&+ \delta_{(c)v} \frac{\partial Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))}{\partial \sigma_1^w} \Big] +
\end{aligned}
$$

$$
+\sum_t \delta_{(c)t} Q_t(\tau,\vec{\xi}(\tau,\vec{\sigma}_1)) \frac{\partial \sigma_1^r(\vec{\xi})}{\partial \xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma}_1)} \cdot
$$

$$
\cdot \frac{\partial}{\partial \sigma_1^r} \big[{}^3e^w_{(k)}(\tau,\vec{\sigma}_1) \frac{\partial \xi^t(\tau,\vec{\sigma}_1)}{\partial \sigma_1^w} T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) \big) \Big). \tag{89}
$$

We have put equal to zero an arbitrary integration constant, namely an arbitrary function $f_{(a)(b)}(\alpha_{(c)},\xi^r,Q_r)$, which would contribute a term $g^r_{(a)} = \frac{1}{2} f_{(a)(b)}\, {}^3e^r_{(b)} = \frac{1}{2Q_r} f_{(a)(b)}(\alpha_{(c)},\xi^s,Q_s)\delta^r_{(b)}$ to ${}^3\tilde{\pi}^r_{(a)}$. Let us remark that the canonical transformation

${}^3e_{(a)r}$, ${}^3\tilde{\pi}^r_{(a)} \mapsto \alpha_{(a)}$, ξ^r, Q_r, $\tilde{\pi}^{\vec{\alpha}}_{(a)}$, $\tilde{\pi}^{\vec{\xi}}_r$, $\tilde{\Pi}^r$

is a point transformation $q^i, p_i \mapsto \tilde{q}^i$, $\tilde{p}_i$ with $q^i = q^i(\tilde{q}^j)$, $p_i(\tilde{q}^j,\tilde{p}_k)$, which is defined modulo a so called trival phase canonical transformation $q^i(\tilde{q}^j)$, $p_i(\tilde{q}^j,\tilde{p}_k) + f_i(\tilde{q}^j)$ with $f_i(\tilde{q}^j) = \partial f(\tilde{q}^j)/\partial \tilde{q}^i$. Therefore, even if we cannot check explicitly the validity of $\{ {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}), {}^3\tilde{\pi}^s_{(b)}(\tau,\vec{\sigma}^{'} \} = 0$ due to the presence of the path-orderings in the expression of the momenta in terms of the new variables, these Poisson brackets imply that $g^r_{(a)}$ is the gradient of a function of $\alpha_{(a)}$, ξ^r, Q_r, so that our choice $f_{(a)(b)} = 0$ amounts to a trivial phase canonical transformation.

Therefore the cotriad and its momentum have the following expression in terms of the new canonical variables [Eqs. (33) and (79) are used]

$$
{}^3e_{(a)r}(\tau,\vec{\sigma}) = {}^3R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\, \frac{\partial \xi^s(\tau,\vec{\sigma})}{\partial \sigma^r}\, \delta_{(b)s}\, Q_s(\tau,\vec{\xi}(\tau,\vec{\sigma})),
$$

$$
\begin{aligned}
{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) &= \frac{1}{2}\, \frac{\delta^r_{(b)}}{Q_r(\tau,\vec{\sigma})}\ \ |\frac{\partial \xi}{\partial \sigma}(\tau,\vec{\sigma})| \cdot \\
&\cdot \Big(\sum_v \Big[2 \sum_s \delta_{(a)s} Q_s(\tau,\vec{\sigma}) \frac{\partial \sigma^s(\vec{\xi})}{\partial \xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\, {}^3R_{(b)(c)}(\alpha_{(e)}(\tau,\vec{\sigma})) - \\
&- [\hat{R}^{(c)}\, {}^3R(\alpha_{(e)}(\tau,\vec{\sigma}))\, \hat{R}^{(b)}]_{(m)(a)} \delta_{(m)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}))\Big]\, \delta_{(c)v} \tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma})) + \\
&+ \int d^3\sigma_1 \sum_v \delta_{(c)v} \tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))\, \frac{|\frac{\partial \xi}{\partial \sigma_1}(\tau,\vec{\sigma}_1)|}{|\frac{\partial \xi}{\partial \sigma}(\tau,\vec{\sigma})|} \cdot \\
&\cdot \Big(\sum_w \frac{\delta_{(k)w}}{Q_w(\tau,\vec{\sigma}_1)} T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) \Big[\delta_{(c)v} \frac{\partial Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))}{\partial \sigma_1^w} + \\
&+ [\hat{R}^{(c)}\, {}^3R(\alpha_{(e)}(\tau,\vec{\sigma}_1))]_{(m)(d)} \delta_{(m)v} Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))^3\omega^{(T)}_{w(d)}(\tau,\vec{\sigma}_1,\alpha_{(e)}(\tau,\vec{\sigma}_1)) \Big] + \\
&+ \sum_t \delta_{(c)t} Q_t(\tau,\vec{\xi}(\tau,\vec{\sigma}_1)) \frac{\partial \sigma_1^s(\vec{\xi})}{\partial \xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma}_1)} \cdot \\
&\cdot \frac{\partial}{\partial \sigma_1^s} \Big[\sum_w \frac{\delta_{(k)w}}{Q_w(\tau,\vec{\sigma}_1)} \frac{\partial \xi^t(\tau,\vec{\sigma}_1)}{\partial \sigma_1^w}\, T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) \Big] \Big) \Big) - \\
&- \frac{1}{2}\, \frac{\delta^r_{(b)}}{Q_r(\tau,\vec{\sigma})} \Big(\epsilon_{(a)(b)(c)} \tilde{\pi}^{\vec{\alpha}}_{(d)}(\tau,\vec{\sigma}) B_{(d)(c)}(\alpha_{(e)}(\tau,\vec{\sigma})) +
\end{aligned}
$$

$$+\int d^3\sigma_1\, T_{(b)(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\sum_w \frac{\delta^w_{(c)}}{Q_w(\tau,\vec{\sigma}_1)}\cdot$$

$$\cdot\;\Big[\frac{\partial\xi^s}{\partial\sigma_1^w}\,\tilde{\pi}^{\vec{\xi}}_s + B_{(d)(f)}(\alpha_{(e)})\,{}^3\omega^{(T)}_{w(f)}(.,\alpha_{(e)})\,\tilde{\pi}^{\vec{\alpha}}_{(d)}\Big](\tau,\vec{\sigma}_1)\;\Big) =$$

$$=\frac{1}{2}\,\frac{\delta^r_{(b)}}{Q_r(\tau,\vec{\sigma})}\;\;|\,\frac{\partial\xi}{\partial\sigma}(\tau,\vec{\sigma})\,|\cdot$$

$$\cdot\;\Big(2\,{}^3R_{(b)(c)}(\alpha_{(e)}(\tau,\vec{\sigma}))\sum_{s,v}\delta_{(a)s}Q_s(\tau,\vec{\sigma})\frac{\partial\sigma^s(\vec{\xi})}{\partial\xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\,\delta_{(c)v}\tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma}))+$$

$$+\int d^3\sigma_1\,\frac{|\frac{\partial\xi}{\partial\sigma_1}(\tau,\vec{\sigma}_1)|}{|\frac{\partial\xi}{\partial\sigma}(\tau,\vec{\sigma})|}\sum_v\delta_{(c)v}\tilde{\Pi}^v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))\cdot$$

$$\cdot\;\Big(\sum_w\frac{\delta_{(c)v}\delta_{(k)w}}{Q_w(\tau,\vec{\sigma}_1)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau)\frac{\partial Q_v(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))}{\partial\sigma_1^w}+$$

$$+\sum_t\delta_{(c)t}Q_t(\tau,\vec{\xi}(\tau,\vec{\sigma}_1))\frac{\partial\sigma_1^s(\vec{\xi})}{\partial\xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma}_1)}\cdot$$

$$\cdot\;\frac{\partial}{\partial\sigma_1^s}\Big[\sum_w\frac{\delta_{(k)w}}{Q_w(\tau,\vec{\sigma}_1)}\frac{\partial\xi^t(\tau,\vec{\sigma}_1)}{\partial\sigma_1^w}\,T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big]\Big)\Big)-$$

$$-\frac{1}{2}\,\frac{\delta^r_{(b)}}{Q_r(\tau,\vec{\sigma})}\Big(\;\epsilon_{(a)(b)(c)}\tilde{\pi}^{\vec{\alpha}}_{(d)}(\tau,\vec{\sigma})B_{(d)(c)}(\alpha_{(e)}(\tau,\vec{\sigma}))+$$

$$+\int d^3\sigma_1\, T_{(b)(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\sum_w \frac{\delta^w_{(c)}}{Q_w(\tau,\vec{\sigma}_1)}\cdot$$

$$\cdot\;\Big[\frac{\partial\xi^s}{\partial\sigma_1^w}\,\tilde{\pi}^{\vec{\xi}}_s + B_{(d)(f)}(\alpha_{(e)})\,{}^3\omega^{(T)}_{w(f)}(.,\alpha_{(e)})\,\tilde{\pi}^{\vec{\alpha}}_{(d)}\Big](\tau,\vec{\sigma}_1)\;\Big) =$$

$$=|_{\alpha_{(a)}=0,\xi^r=\sigma^r,\tilde{\pi}^{\vec{\alpha}}_{(a)}=\tilde{\pi}^{\vec{\xi}}_r=0}\quad {}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}). \tag{90}$$

and from Eqs.(57) and (84) of I we get

$$\begin{aligned}
{}^3K_{rs} &= \sum_u\frac{\epsilon Q_rQ_sQ_u}{4kQ_1Q_2Q_3}\,{}^3G_{o(a)(b)(c)(d)}\delta_{(a)r}\delta_{(b)s}\delta_{(c)u}\,{}^3\tilde{\pi}^u_{(d)},\\
{}^3K &= -\frac{\epsilon}{2kQ_1Q_2Q_3}\,{}^3\tilde{\Pi} = -\frac{\epsilon}{4kQ_1Q_2Q_3}\sum_r\delta_{(a)r}Q_r\,{}^3\tilde{\pi}^r_{(a)},\\
{}^2\tilde{\Pi}^{rs} &= \epsilon kQ_1Q_2Q_3({}^3K^{rs}-Q_r^2\delta^{rs}\,{}^3K) = \frac{1}{4}\Big[\frac{\delta^r_{(a)}\,{}^3\tilde{\pi}^s_{(a)}}{Q_r}+\frac{\delta^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)}}{Q_s}\Big].
\end{aligned} \tag{91}$$

Due to the presence of the Green function it is not possible to rewrite the final expression of ${}^3\tilde{\pi}^r_{(a)}$ explicitly in the form of Eq.(60).

However, the functions ${}^3\Gamma^u_{rs}$, ${}^3R_{rsuv}$, ${}^3\omega_{r(a)}$, ${}^3\Omega_{rs(a)}$ and ,by using Eqs.(57) of I, ${}^3K_{rs}$ [and also the metric ADM momentum ${}^3\tilde{\Pi}^{rs}$ of Eq.(84) of I and the Weyl-Schouten ${}^3C_{rsu}$ and Cotton-York ${}^3\mathcal{H}_{rs}$ tensors defined after Eq.(9) of I] may now be expressed in terms of $\alpha_{(a)}$, $\tilde{\pi}^{\vec{\alpha}}_{(a)}$, ξ^r, $\tilde{\pi}^{\vec{\xi}}_r$, Q_r, $\tilde{\Pi}^r$, and then Eqs.(38), (39), (40), (43), (46), (47), (A1), (A4), (A5), (A6) of

I, allow to reconstruct the functions ${}^4g_{AB}$, ${}^4E^{(\alpha)}_{\mu}$, ${}^4\Gamma^{\alpha}_{\beta\gamma}$, ${}^4R_{\mu\nu\alpha\beta}$, ${}^4\omega_{\mu(\alpha)(\beta)}$, ${}^4\Omega_{\mu\nu(\alpha)(\beta)}$, ${}^4C_{\mu\nu\alpha\beta}$, in terms of the canonical basis $\tilde{\lambda}_A$, $\tilde{\pi}^A \approx 0$, $\tilde{\lambda}_{AB}$, $\tilde{\pi}^{AB} \approx 0$, n, $\tilde{\pi}^n \approx 0$, $n_{(a)}$, $\tilde{\pi}^{\vec{n}}_{(a)} \approx 0$, $\varphi_{(a)}$, $\tilde{\pi}^{\vec{\varphi}}_{(a)} \approx 0$, $\alpha_{(a)}$, $\tilde{\pi}^{\vec{\alpha}}_{(a)} \approx 0$, ξ^r, $\tilde{\pi}^{\vec{\xi}}_r \approx 0$, Q_r, $\tilde{\Pi}^r$. In the new basis only the superhamiltonian constraint of Eq.(61) of I is left. Instead the inverse canonical transformation cannot be computed explicitly till when one does not understand how to solve Eqs.(63).

V. A NEW CANONICAL BASIS AND THE SUPERHAMILTONIAN CONSTRAINT.

Let us study the reduced phase space spanned by the canonical coordinates $\tilde{\lambda}_A$, $\tilde{\pi}^A \approx 0$, $\tilde{\lambda}_{AB}$, $\tilde{\pi}^{AB} \approx 0$, n, $\tilde{\pi}^n \approx 0$, $n_{(a)}$, $\tilde{\pi}^{\vec{n}}_{(a)} \approx 0$, $\varphi_{(a)}$, $\tilde{\pi}^{\vec{\varphi}}_{(a)} \approx 0$ (for the spacetime description), and Q_r, $\tilde{\Pi}^r$ (for the superspace of 3-geometries) obtained by adding the gauge-fixing constraints $\varphi_{(a)}(\tau,\vec{\sigma}) \approx 0$ [their time constancy implies $\lambda^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) \approx 0$], $\alpha_{(a)}(\tau,\vec{\sigma}) \approx 0$, $\xi^r(\tau,\vec{\sigma}) \approx \sigma^r$ and by going to Dirac brackets. This means to restrict the Cauchy data of cotriads on Σ_τ by eliminating the gauge degrees of freedom of boosts, rotations and space pseudodiffeomorphisms, i.e. by restricting ourselves to 3-orthogonal coordinates on Σ_τ and by having made the choice of the Σ_τ-adapted tetrads ${}^4_{(\Sigma_\tau)}\check{\tilde{E}}^A_{(\alpha)}$ [see Eqs.(39), (40) of I rewritten in terms of the Dirac observables ${}^3\hat{e}^r_{(a)}$ dual to ${}^3\hat{e}_{(a)r}$] as the reference nongeodesic congruence of timelike "nonrotating" observers with 4-velocity field $l^A(\tau,\vec{\sigma})$.

By remembering Eqs.(43) and (50), the Dirac brackets are strongly equal to

$$
\begin{aligned}
\{A(\tau,\vec{\sigma})\ ,\ & B(\tau,\vec{\sigma}^{'})\}^* = \{A(\tau,\vec{\sigma}), B(\tau,\vec{\sigma}^{'})\} + \\
&+ \int d^3\sigma_1 \Big[\{A(\tau,\vec{\sigma}), \alpha_{(a)}(\tau,\vec{\sigma}_1)\}\{\tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\} - \\
&- \{A(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{\alpha}}_{(a)}(\tau,\vec{\sigma}_1)\}\{\alpha_{(a)}(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\} + \\
&+ \{A(\tau,\vec{\sigma}), \xi^r(\tau,\vec{\sigma}_1)\}\{\tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\} - \\
&- \{A(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma}_1)\}\{\xi^r(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\}\Big] \equiv \\
&\equiv \{A(\tau,\vec{\sigma}), B(\tau,\vec{\sigma}^{'})\} + \\
&+ \int d^3\sigma_1 \Big(\Big[\{A(\tau,\vec{\sigma}), \alpha_{(a)}(\tau,\vec{\sigma}_1)\}\{{}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\} - \\
&- \{A(\tau,\vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}_1)\}\{\alpha_{(a)}(\tau,\vec{\sigma}_1) B(\tau,\vec{\sigma}^{'})\}\Big] \cdot \\
&\cdot\ A_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}_1)) + \frac{\partial \sigma_1^s(\vec{\xi})}{\partial \xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma}_1)} \cdot \\
&\cdot\ \Big[\{A(\tau,\vec{\sigma}), \xi^r(\tau,\vec{\sigma}_1)\}\{{}^3\tilde{\Theta}_s(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\} - \\
&- \{A(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_s(\tau,\vec{\sigma}_1)\}\{\xi^r(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\}\Big] + \\
&+ \frac{\partial \sigma_1^s(\vec{\xi})}{\partial \xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma}_1)} A_{(b)(a)}(\alpha_{(e)}(\tau,\vec{\sigma}_1)) \frac{\partial \alpha_{(a)}(\tau,\vec{\sigma}_1)}{\partial \sigma_1^s} \cdot \\
&\cdot\ \Big[\{A(\tau,\vec{\sigma}), \xi^r(\tau,\vec{\sigma}_1)\}\{{}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\} - \\
&- \{A(\tau,\vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}_1)\}\{\xi^r(\tau,\vec{\sigma}_1), B(\tau,\vec{\sigma}^{'})\}\Big] \Big).
\end{aligned}
\qquad (92)
$$

Since the variables $\alpha_{(a)}(\tau,\vec{\sigma})$, $\xi^r(\tau,\vec{\sigma})$, are not known as explicit functions of the cotriads, these Dirac brackets can be used only implicitly. As it will be shown in Ref. [6], we must have $\alpha_{(a)}(\tau,\vec{\sigma}) \rightarrow O(r^{-(1+\epsilon)})$ and $\xi^r(\tau,\vec{\sigma}) \rightarrow \sigma^r + O(r^{-\epsilon})$ for $r \rightarrow \infty$ to preserve Eqs.(6).

We have seen in Section III that the differential geometric description for rotations already showed that the restriction to the identity cross section $\alpha_{(a)}(\tau,\vec{\sigma}) = 0$ implied also $\partial_r \alpha_{(a)}(\tau,\vec{\sigma}) = 0$; we also have $A_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))|_{\alpha=0} = 0$. When we add the gauge-fixings

$\alpha_{(a)}(\tau,\vec{\sigma}) \approx 0$, the derivatives of all orders of $\alpha_{(a)}(\tau,\vec{\sigma})$ weakly vanish at $\alpha_{(a)}(\tau,\vec{\sigma}) = 0$. Similarly, it can be shown that, if we have the pseudodiffeomorphism $\vec{\xi}(\tau,\vec{\sigma}) = \vec{\sigma} + \hat{\vec{\xi}}(\tau,\vec{\sigma})$ so that for $\vec{\xi}(\tau,\vec{\sigma}) \rightarrow \vec{\sigma}$ we have $\hat{\vec{\xi}}(\tau,\vec{\sigma}) \rightarrow \delta\vec{\sigma}(\tau,\vec{\sigma})$, then the quantities ${}^3e_{(a)r}(\tau,\vec{\sigma})$, $\partial_r\, {}^3e_{(a)s}(\tau,\vec{\sigma})$, ${}^3\omega_{r(a)}(\tau,\vec{\sigma})$, ${}^3\Omega_{rs(a)}(\tau,\vec{\sigma})$, become functions only of $Q_r(\tau,\vec{\sigma})$ for $\vec{\xi}(\tau,\vec{\sigma}) \rightarrow \vec{\sigma}$ and $\alpha_{(a)}(\tau,\vec{\sigma}) \rightarrow 0$ only if we have the following behaviour of the parameters $\xi^r(\tau,\vec{\sigma})$

$$
\begin{aligned}
&\frac{\partial \delta\sigma^r(\tau,\vec{\sigma})}{\partial\sigma^s}|_{\vec{\xi}=\vec{\sigma}} = 0 \Rightarrow \frac{\partial \xi^r(\tau,\vec{\sigma})}{\partial\sigma^s}|_{\vec{\xi}=\vec{\sigma}} = \delta^r_s, \quad \frac{\partial^2 \xi^r(\tau,\vec{\sigma})}{\partial\sigma^s\partial\sigma^u}|_{\vec{\xi}=\vec{\sigma}} = 0,\\
&\frac{\partial^2 \delta\sigma^r(\tau,\vec{\sigma})}{\partial\sigma^u\partial\sigma^v}|_{\vec{\xi}=\vec{\sigma}} = 0 \Rightarrow [\frac{\partial}{\partial\sigma^u}\frac{\partial \sigma^r(\tau,\vec{\sigma})}{\partial\xi^v}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}]|_{\vec{\xi}=\vec{\sigma}} = 0,\\
&\frac{\partial^3 \delta\sigma^r(\tau,\vec{\sigma})}{\partial\sigma^s\partial\sigma^u\partial\sigma^v}|_{\vec{\xi}=\vec{\sigma}} = 0 \Rightarrow [\frac{\partial^2}{\partial\sigma^u\partial\sigma^v}\frac{\partial \sigma^r(\vec{\xi})}{\partial\xi^s}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}]|_{\vec{\xi}=\vec{\sigma}} = 0.
\end{aligned}
\tag{93}
$$

These conditions should be satisfied by the parameters of pseudodiffeomorphisms near the identity, i.e. near the chart chosen as reference chart (the 3-orthogonal one in this case). With the gauge-fixings $\xi^r(\tau,\vec{\sigma}) \approx \vec{\sigma}$ all these properties are satisfied.

By using the Dirac Hamiltonian (69)

$$\hat{H}_{(D)ADM} = \int d^3\sigma [n\hat{\mathcal{H}} - \tilde{n}^r \tilde{\pi}^{\vec{\xi}}_r + \lambda_n \tilde{\pi}^n + \lambda^{\vec{n}}_{(a)} \tilde{\pi}^{\vec{\varphi}}_{(a)} + \tilde{\mu}_{(a)} \tilde{\pi}^{\vec{\alpha}}_{(a)}](\tau,\vec{\sigma}) + \zeta_A(\tau)\tilde{\pi}^A(\tau) + \zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau)$$

with $\tilde{n}^r = n_{(a)}\, {}^3e^s_{(a)} \frac{\partial\xi^r}{\partial\sigma^s} = n_u\, {}^3g^{uv} \frac{\partial\xi^r}{\partial\sigma^v}$, the time constancy of the gauge fixings gives

$\partial_\tau \alpha_{(a)} \overset{\circ}{=} \tilde{\mu}_{(a)} \approx 0$ and $\partial_\tau[\xi^r - \sigma^r] \overset{\circ}{=} \tilde{n}^r = n_{(a)}\, {}^3e^s_{(a)} \frac{\partial\xi^r}{\partial\sigma^s} \approx 0$,

so that we get the three new constraints $n_{(a)}(\tau,\vec{\sigma}) \approx 0$ implying the vanishing of the part of the shift vector associated with proper gauge transformations [and $N_{(a)} = N_{(as)\,(a)}$, see Eq.(4)]. Then we have $\partial_\tau n_{(a)} \overset{\circ}{=} \lambda^{\vec{n}}_{(a)} \approx 0$. Now $n_{(a)}(\tau,\vec{\sigma}) \approx 0$ implies $n^r(\tau,\vec{\sigma}) \approx 0$ and, from Eqs.(72),

$$ds^2 = \epsilon([N_{(as)} + n]^2 - \sum_u N^2_{(as)u}/Q^2_u)(d\tau)^2 - 2\epsilon N_{(as)r} d\tau d\sigma^r - \epsilon \sum_u Q^2_u (d\sigma^u)^2.$$

If we would add the extra gauge-fixings $N_{(as)r} \approx 0$, this would be the definition of "synchronous" coordinates in M^4, whose problem is the tendency to develop coordinate singularities in short times [53,54] [see Ref. [55] for the problems of the fixation of N and N^r in ADM metric gravity (coordinate conditions to rebuild spacetime) and for the origin of the coordinates used in numerical gravity (see Ref. [56] for a recent review of it)].

However, as we shall see in the next paper [6], these results will not be valid after the addition to the Dirac Hamiltonian of the surface terms needed to make it differentiable.

Since, as already said, a change of coordinate chart with a space pseudodiffeomorphism implies the redefinition of the functions $\vec{\xi}(\tau,\vec{\sigma})$, we should explore systematically the effect of other gauge-fixings of the type $\vec{\xi}(\tau,\vec{\sigma}) - \vec{f}(\tau,\vec{\sigma}) \approx 0$ for arbitrary vector functions $\vec{f}$ [so that $\partial_\tau(\xi^r - f^r) \overset{\circ}{=} n_{(a)}\, {}^3e^s_{(a)} \frac{\partial\xi^r}{\partial\sigma^s} - \partial_\tau f^r \approx 0$, which implies $n_{(a)} \approx {}^3e_{(a)r} \frac{\partial\sigma^r}{\partial\xi^s}\partial_\tau f^s$ or $n_u\, {}^3g^{uv}\partial_v f^r \approx \partial_\tau f^r$], which describes the "residual gauge freedom" of going from a 3-orthogonal gauge "in M^4" to another one [the allowed canonical transformations $Q_r, \tilde{\Pi}^r \mapsto \check{Q}^r(Q), \check{\tilde{\Pi}}^r(Q,\tilde{\Pi})$ which leave ${}^3\hat g_{rs}$ diagonal]. Let us remark that a redefinition of the functions $\vec \xi(\tau ,\vec \sigma)$ also implies a redefinition of the angles $\alpha_{(a)}(\tau ,\vec \sigma)$ [rotations and pseudodiffeomorphisms do not commute]: therefore this "residual gauge freedom" allows to go from a 3-orthogonal gauge A_1 to another one A_2 "rotating" with respect to A_1 [so that for instance we could get $n_r(\tau ,\vec \sigma)=2\epsilon_{rst}\sigma^s\omega^t$: for the observer in $(\tau ,\vec \sigma)$ in A_1 the triad ${}^3\hat e^r_{(a)}(\tau ,\vec \sigma)$ would be Fermi-Walker transported along his worldline, while in A_2 it would rotate with angular velocity $\vec \omega$ with respect to the one in A_1 [57,63]].

The Dirac Hamiltonian reduces to

$$H_{(D)ADM,R} \equiv \int d^3\sigma [n \hat{\cal H}_R+\lambda_n\tilde \pi^n](\tau ,\vec \sigma)+\zeta_A(\tau)\tilde \pi^A(\tau)+\zeta_{AB}(\tau)\tilde \pi^{AB}(\tau),$$

where $\hat{\cal H}_R$ is the reduced superhamiltonian constraint.

This amounts to the Schwinger time gauge: Eqs. (46), (40) of I imply for the cotetrad ${}^4E^{(\alpha)}_A={}^4_{(\Sigma)}\check{\tilde E}^{(\alpha)}_A$ with ${}^4_{(\Sigma)}\check{\tilde E}^{(o)}_{\tau}=N_{(as)}+n$, ${}^4_{(\Sigma)}\check{\tilde E}^{(o)}_r=0$, ${}^4_{(\Sigma)}\check{\tilde E}^{(a)}_{\tau}=N_{(as)(a)}$, ${}^4_{(\Sigma)}\check{\tilde E}^{(a)}_r={}^3\hat e_{(a)r}$.

At the level of Dirac brackets the constraints $\hat {\cal H}_{(a)}\approx 0$ (or ${}^3\tilde \Theta_r\approx 0$) and ${}^3\tilde M_{(a)}\approx 0$ [and also the derived ADM constraints ${}^3\tilde \Pi^{rs}{}_{|s}\approx 0$ as shown in Section V of I] hold strongly, so that the reduced quantities ${}^3\hat e_{(a)r}$ and ${}^3\hat{\tilde \pi}^r_{(a)}$ [which now describe only three pairs of conjugate variables in each point of Σ_{τ}] must obey

$$\begin{array}{l}
{}^3\hat e_{(a)r}\, {}^3\hat{\tilde \pi}^r_{(b)}-{}^3\hat e_{(b)r}\, {}^3\hat{\tilde \pi}^r_{(a)}\equiv 0,\qquad \partial_r\, {}^3\hat{\tilde \pi}^r_{(a)}-\epsilon_{(a)(b)(c)}\, {}^3\hat \omega_{r(b)}\, {}^3\hat{\tilde \pi}^r_{(c)}\equiv 0,\\
or\quad {}^3\hat{\tilde \pi}^s_{(a)}\partial_r\, {}^3\hat e_{(a)s}-\partial_s({}^3\hat e_{(a)r}\, {}^3\hat{\tilde \pi}^s_{(a)})\equiv 0,\\
or\quad \partial_s\, {}^3\hat{\tilde \Pi}^{rs}+{}^3\hat \Gamma^r_{su}\, {}^3\hat{\tilde \Pi}^{su}\equiv 0.
\end{array}$$

Therefore, the ADM momentum ${}^3\hat{\tilde \Pi}^{rs}$ is strongly transverse, ${}^3\hat{\tilde \Pi}^{rs}\equiv {}^3\hat{\tilde \Pi}^{rs}_t$, and, according to the result (C4) of Appendix C, can be written as ${}^3\hat{\tilde \Pi}^{rs}_t={}^3\hat{\tilde \Pi}^{rs}_{TT}+{}^3\hat{\tilde \Pi}^{rs}_{Tr,t}$ with both the terms transverse and the first one traceless. Since now ${}^3\hat{\tilde \Pi}^{rs}_t$ contains only 3 independent degrees of freedom [the ${}^3\tilde \Pi^r(\tau ,\vec \sigma)$], we see that ${}^3\hat{\tilde \Pi}^{rs}_{TT}$ should describe the spin-two wave part of the ADM momentum, while ${}^3\hat{\tilde \Pi}^{rs}_{Tr,t}$ should describe the mean extrinsic curvature through its trace. However, Eq.(84) of I does not imply $\{ {}^3\hat{\tilde \Pi}^{rs}(\tau ,\vec \sigma), {}^3\hat{\tilde \Pi}^{uv}(\tau ,{\vec \sigma}_1)\}^{*}=0$ at the level of these Dirac brackets, since ${}^3\tilde \Pi^{rs}(\tau ,\vec \sigma)$ does not commute with the supermomentum constraints [one would get $\{ {}^3\hat{\tilde \Pi}^{rs}(\tau ,\vec \sigma), {}^3\hat{\tilde \Pi}^{uv}(\tau ,{\vec \sigma}_1)\}^{*}=0$ if these would be the Dirac brackets only with respect to the second class pairs $\tilde \pi^{\vec \alpha}_{(a)}\approx 0$, $\alpha_{(a)}\approx 0$, in accord with Section V of I].

Some algebraic calculations for $\xi^r(\tau ,\vec \sigma)\rightarrow \sigma^r$ give [to get the expressions with $n_r\not= 0$, replace $N_{(as)r}$ with $N_{(as)r}+n_r$]

$$\begin{array}{l}
{}^3e_{(a)r}\mapsto {}^3\hat e_{(a)r}=\delta_{(a)r}Q_r,\\
{}^3e^r_{(a)}\mapsto {}^3\hat e^r_{(a)}={\displaystyle\frac{\delta^r_{(a)}}{Q_r}},\\
{}^3e=det\, |{}^3e_{(a)r}| =\sqrt{\gamma}\mapsto {}^3\hat e=\sqrt{\hat \gamma}=Q_1Q_2Q_3,\\
{}^3g_{rs}={}^3e_{(a)r}\, {}^3e_{(a)s}\mapsto {}^3\hat g_{rs}=\delta_{rs}Q^2_r,
\end{array}$$

$$
\begin{aligned}
&{}^3g^{rs} = {}^3e^r_{(a)}\, {}^3e^s_{(a)} \mapsto {}^3\hat{g}^{rs} = \delta^{rs}/Q^2_r,\\
&ds^2 \mapsto d\hat{s}^2 = \epsilon\Big[(N_{(as)}+n)^2 - \sum_u \frac{N^2_{(as)u}}{Q^2_u}\Big](d\tau)^2 -\\
&- 2\epsilon N_{(as)r} d\tau d\sigma^r - \epsilon \sum_u Q^2_u (d\sigma^u)^2 = \epsilon\Big[(N_{(as)}+n)^2(d\tau)^2 -\\
&-\delta_{uv}(Q_u d\sigma^u + \frac{N_{(as)u}}{Q_u} d\tau)(Q_v d\sigma^v + \frac{N_{(as)v}}{Q_v} d\tau)\Big],\\
&{}^3\Gamma^r_{uv} \mapsto {}^3\hat{\Gamma}^r_{uv} = \delta^r_u \frac{\partial_v Q_u}{Q_u} + \delta^r_v \frac{\partial_u Q_v}{Q_v} - \delta_{uv}\delta^r_s \frac{Q_u \partial_s Q_u}{Q^2_s} =\\
&= -\delta_{uv}\delta^r_s (\frac{Q_u}{Q_s})^2 \partial_s ln\, Q_u + \delta^r_u \partial_v ln\, Q_u + \delta^r_v \partial_u ln\, Q_v,\\
&{}^3\omega_{r(a)} \mapsto {}^3\hat{\omega}_{r(a)} = \epsilon_{(a)(b)(c)}\delta_{(b)r}\delta_{(c)u}\frac{\partial_u Q_r}{Q_u} = \epsilon_{(a)(b)(c)}\delta_{(b)r}\delta_{(c)u}\frac{Q_r}{Q_u}\partial_u ln\, Q_r.
\end{aligned} \tag{94}
$$

The expressions for ${}^3\Omega_{rs(a)}$, ${}^3R_{rsuv}$, ${}^3R_{rs}$, ${}^3R$, will be given in Appendix D after a final canonical transformation.

Moreover, from Eq.(90) we have

$$
{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) \mapsto {}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}) = \int d^3\sigma_1\, \mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|Q]\, \tilde{\Pi}^s(\tau,\vec{\sigma}_1),
$$

$$
\begin{aligned}
\mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|Q] &= \delta^r_{(a)}\delta^r_s\delta^3(\vec{\sigma},\vec{\sigma}_1) + \mathcal{T}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|Q],\\
\mathcal{T}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|Q] &= \frac{\delta^r_{(b)}}{2Q_r(\tau,\vec{\sigma})}\Big[\sum_{w\neq s}\frac{\delta_{(k)w}}{Q_w(\tau,\vec{\sigma}_1)}\frac{\partial Q_s(\tau,\vec{\sigma}_1)}{\partial\sigma^w_1} T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) +\\
&+ \delta_{(k)s}\frac{\partial}{\partial\sigma^s_1} T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big],\\
\delta^r_{(b)} T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) &= Q_r(\tau,\vec{\sigma}) d^r_{\gamma_{PP_1}}(\vec{\sigma},\vec{\sigma}_1)\Big(P_{\gamma_{PP_1}}\, e^{\int^{\vec{\sigma}}_{\vec{\sigma}_1} d\sigma^w_2\, {}^3\hat{\omega}_{w(c)}\hat{R}^{(c)}}\Big)_{(a)(k)} +\\
&+ \sum_u \delta_{(a)u} Q_u(\tau,\vec{\sigma}) d^u_{\gamma_{PP_1}}(\vec{\sigma},\vec{\sigma}_1)\delta^r_{(b)}\Big(P_{\gamma_{PP_1}}\, e^{\int^{\vec{\sigma}}_{\vec{\sigma}_1} d\sigma^w_2\, {}^3\hat{\omega}_{w(c)}\hat{R}^{(c)}}\Big)_{(b)(k)}.
\end{aligned} \tag{95}
$$

so that we have

$$
\begin{aligned}
&{}^3K_{rs} \mapsto {}^3\hat{K}_{rs} = \frac{\epsilon}{4k}\frac{Q_rQ_s}{Q_1Q_2Q_3}\sum_u \Big(\delta_{ru}\delta_{(a)s} + \delta_{su}\delta_{(a)r} - \delta_{rs}\delta_{(a)u}\Big) Q_u\, {}^3\hat{\tilde{\pi}}^u_{(a)},\\
&{}^3K \mapsto {}^3\hat{K} = {}^3\hat{g}^{rs}\, {}^3\hat{K}_{rs} = -\frac{\epsilon}{4k}\sum_u \delta_{(a)u}\frac{Q_u}{Q_1Q_2Q_3}\, {}^3\hat{\tilde{\pi}}^u_{(a)},\\
&{}^3\tilde{\Pi}^{rs} \mapsto {}^3\hat{\tilde{\Pi}}^{rs} = \frac{1}{4}\Big[\frac{\delta^r_{(a)}}{Q_r}\, {}^3\hat{\tilde{\pi}}^s_{(a)} + \frac{\delta^s_{(a)}}{Q_s}\, {}^3\hat{\tilde{\pi}}^r_{(a)}\Big] \equiv {}^3\hat{\tilde{\Pi}}^{rs}_t = {}^3\hat{\tilde{\Pi}}^{rs}_{TT} + {}^3\hat{\tilde{\Pi}}^{rs}_{Tr,t},\\
&{}^3\tilde{\Pi} = {}^3g_{rs}\, {}^3\tilde{\Pi}^{rs} \mapsto {}^3\hat{\tilde{\Pi}} = -2\epsilon k\, Q_1Q_2Q_3\, {}^3\hat{K} = \frac{1}{2}\sum_r Q_r\delta_{(a)r}\, {}^3\hat{\tilde{\pi}}^r_{(a)},
\end{aligned}
$$

$$
{}^3\tilde{\Pi}_{(a)(b)} = {}^3e_{(a)r}\, {}^3e_{(b)s}\, {}^3\tilde{\Pi}^{rs} \mapsto {}^3\hat{\tilde{\Pi}}_{(a)(b)} = \frac{1}{4}\sum_r Q_r \Big[\delta_{(a)r}\, {}^3\hat{\tilde{\pi}}^r_{(b)} + \delta_{(b)r}\, {}^3\hat{\tilde{\pi}}^r_{(a)}\Big]. \tag{96}
$$

The determination of the gravitomagnetic potential $W^r(\tau,\vec{\sigma})$, see Appendix C, by solving the elliptic equations associated with the supermomentum constraints in the conformal approach to metric gravity, has been replaced here by the determination of the kernel $\mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}^{'};\tau|Q]$ connecting the old momenta ${}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma})$ to the new canonical ones $\tilde{\Pi}^r(\tau,\vec{\sigma})$.

The reduced superhamiltonian constraint becomes [in the last line Eq(7) is used; $k = c^3/16\pi G$ with G the Newton constant]

$$
\begin{aligned}
&\hat{\mathcal{H}}(\tau,\vec{\sigma}) = \epsilon\Big[k^3 e\, \epsilon_{(a)(b)(c)}\, {}^3e^r_{(a)}\, {}^3e^s_{(b)}\, {}^3\Omega_{rs(c)} - \\
-\, &\frac{1}{8k\, {}^3e}\, {}^3G_{o(a)(b)(c)(d)}\, {}^3e_{(a)r}\, {}^3\tilde{\pi}^r_{(b)}\, {}^3e_{(c)s}\, {}^3\tilde{\pi}^s_{(d)}\Big](\tau,\vec{\sigma}) \\
&\mapsto \hat{\mathcal{H}}_R(\tau,\vec{\sigma}) = \epsilon\Big[kQ_1Q_2Q_3\epsilon_{(a)(b)(c)}\sum_{r,s}\frac{\delta_{(a)r}\delta_{(b)s}}{Q_rQ_s}\, {}^3\hat{\Omega}_{rs(c)} - \\
-\, &\frac{1}{8k\, Q_1Q_2Q_3}\sum_{r,s}\Big(\delta_{rs}\delta_{(a)(b)} + \delta_{(a)r}\delta_{(b)s} - \delta_{(a)s}\delta_{(b)r}\Big)\, Q_r\, {}^3\hat{\tilde{\pi}}^r_{(b)}\, Q_s\, {}^3\hat{\tilde{\pi}}^s_{(a)}\Big](\tau,\vec{\sigma}) = \\
=\, &-\sum_{r,s}\epsilon\Big(\frac{Q_rQ_s}{8k\, Q_1Q_2Q_3}\Big)(\tau,\vec{\sigma}) \int d^3\sigma_1 d^3\sigma_2\, \tilde{\Pi}^u(\tau,\vec{\sigma}_1)\mathcal{K}^r_{(b)u}(\vec{\sigma},\vec{\sigma}_1;\tau|Q]\cdot \\
\cdot\, &\Big(\delta_{rs}\delta_{(a)(b)} + \delta_{(a)r}\delta_{(b)s} - \delta_{(a)s}\delta_{(b)r}\Big)\, \mathcal{K}^s_{(a)v}(\vec{\sigma},\vec{\sigma}_2;\tau|Q]\, \tilde{\Pi}^v(\tau,\vec{\sigma}_2) + \\
+\, &\epsilon\Big(kQ_1Q_2Q_3\, \epsilon_{(a)(b)(c)}\sum_{r,s}\frac{\delta_{(a)r}\delta_{(b)s}}{Q_rQ_s}\, {}^3\hat{\Omega}_{rs(c)}\Big)(\tau,\vec{\sigma}) = \\
=\, &-\epsilon\Big(\frac{1}{8k\, Q_1Q_2Q_3}\Big)(\tau,\vec{\sigma})\Big(\, \Big[2\sum_u(Q_u\tilde{\Pi}^u)^2 - (\sum_u Q_u\tilde{\Pi}^u)^2\Big](\tau,\vec{\sigma}) + \\
+\, &2\sum_{r,s} Q_r(\tau,\vec{\sigma})Q_s(\tau,\vec{\sigma})\tilde{\Pi}^r(\tau,\vec{\sigma})(2\delta_{rs}-1)\delta_{(a)s}\cdot \\
\cdot\, &\int d^3\sigma_1\, \mathcal{T}^s_{(a)v}(\vec{\sigma},\vec{\sigma}_1;\tau|Q]\tilde{\Pi}^v(\tau,\vec{\sigma}_1) + \\
+\, &\sum_{r,s} Q_r(\tau,\vec{\sigma})Q_s(\tau,\vec{\sigma})\int d^3\sigma_1 d^3\sigma_2\, \tilde{\Pi}^u(\tau,\vec{\sigma}_1)\mathcal{T}^r_{(b)u}(\vec{\sigma},\vec{\sigma}_1;\tau|Q]\cdot \\
\cdot\, &\Big(\delta_{rs}\delta_{(a)(b)} + \delta_{(a)r}\delta_{(b)s} - \delta_{(a)s}\delta_{(b)r}\Big)\mathcal{T}^s_{(a)v}(\vec{\sigma},\vec{\sigma}_2;\tau|Q]\, \tilde{\Pi}^v(\tau,\vec{\sigma}_2)\, \Big) + \\
+\, &\epsilon\Big[kQ_1Q_2Q_3\epsilon_{(a)(b)(c)}\sum_{r,s}\frac{\delta_{(a)r}\delta_{(b)s}}{Q_rQ_s}\, {}^3\hat{\Omega}_{rs(c)}[Q]\,\Big](\tau,\vec{\sigma}) \approx 0, \\
&\{\hat{\mathcal{H}}_R(\tau,\vec{\sigma}), \hat{\mathcal{H}}_R(\tau,\vec{\sigma}^{'})\}^* \equiv \{\hat{\mathcal{H}}(\tau,\vec{\sigma}), \hat{\mathcal{H}}(\tau,\vec{\sigma}^{'})\}^* \equiv \\
&\equiv -\frac{\partial}{\partial\sigma^s}\Big[\frac{\partial\sigma^s(\vec{\xi})}{\partial\xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma})}\, \{\xi^r(\tau,\vec{\sigma}), \hat{\mathcal{H}}(\tau,\vec{\sigma}^{'})\}\, \hat{\mathcal{H}}_R(\tau,\vec{\sigma})\Big] + \\
&+\frac{\partial}{\partial\sigma^{'s}}\Big[\frac{\partial\sigma^{'s}(\vec{\xi})}{\partial\xi^r}|_{\vec{\xi}=\vec{\xi}(\tau,\vec{\sigma}^{'})}\, \{\xi^r(\tau,\vec{\sigma}^{'}), \hat{\mathcal{H}}(\tau,\vec{\sigma})\}\, \hat{\mathcal{H}}_R(\tau,\vec{\sigma}^{'})\Big] \approx 0.
\end{aligned} \tag{97}
$$

The constraint is no more an algebraic relation among the final variables, but rather an integrodifferential equation for one of them.

Let us now consider a new canonical transformation from the basis $Q_u(\tau,\vec{\sigma})$, $\tilde{\Pi}^u(\tau,\vec{\sigma})$ to a new basis $q_u(\tau,\vec{\sigma})$, $\rho_u(\tau,\vec{\sigma})$ defined in the following way

$$
\begin{aligned}
q_u(\tau,\vec{\sigma}) &= ln\, Q_u(\tau,\vec{\sigma}),\\
\rho_u(\tau,\vec{\sigma}) &= Q_u(\tau,\vec{\sigma})\, \tilde{\Pi}^u(\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
&\{q_u(\tau,\vec{\sigma}), \rho_v(\tau,\vec{\sigma}^{'})\} = \delta_{uv}\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&Q_u(\tau,\vec{\sigma}) = e^{q_u(\tau,\vec{\sigma})}, \qquad \tilde{\Pi}^u(\tau,\vec{\sigma}) = e^{-q_u(\tau,\vec{\sigma})}\, \rho_u(\tau,\vec{\sigma}).
\end{aligned}
\tag{98}
$$

It is convenient to make one more canonical transformation, like for the determination of the center of mass of a particle system [5], to the following new set

$$
\begin{aligned}
q(\tau,\vec{\sigma}) &= \frac{1}{3}\sum_u q_u(\tau,\vec{\sigma}) = \frac{1}{3}\sum_u ln\, Q_u(\tau,\vec{\sigma}),\\
r_{\bar{a}}(\tau,\vec{\sigma}) &= \sqrt{3}\sum_u \gamma_{\bar{a}u} q_u(\tau,\vec{\sigma}) = \sqrt{3}\sum_u \gamma_{\bar{a}u} ln\, Q_u(\tau,\vec{\sigma}), \qquad \bar{a}=1,2,\\
\rho(\tau,\vec{\sigma}) &= \sum_u \rho_u(\tau,\vec{\sigma}) = \sum_u [Q_u\tilde{\Pi}^u](\tau,\vec{\sigma}),\\
\pi_{\bar{a}}(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{3}}\sum_u \gamma_{\bar{a}u}\rho_u(\tau,\vec{\sigma}) = \frac{1}{\sqrt{3}}\sum_u \gamma_{\bar{a}u}[Q_u\tilde{\Pi}^u](\tau,\vec{\sigma}), \qquad \bar{a}=1,2,\\
&\{q(\tau,\vec{\sigma}),\rho(\tau,\vec{\sigma}^{'})\} = \delta^3(\vec{\sigma},\vec{\sigma}^{'}), \qquad \{r_{\bar{a}}(\tau,\vec{\sigma}),\pi_{\bar{b}}(\tau,\vec{\sigma}^{'})\} = \delta_{\bar{a}\bar{b}}\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
q_u(\tau,\vec{\sigma}) &= q(\tau,\vec{\sigma}) + \frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u} r_{\bar{a}}(\tau,\vec{\sigma}), \qquad Q_u(\tau,\vec{\sigma}) = e^{q_u(\tau,\vec{\sigma})},\\
\rho_u(\tau,\vec{\sigma}) &= \frac{1}{3}\rho(\tau,\vec{\sigma}) + \sqrt{3}\sum_{\bar{a}}\gamma_{\bar{a}u}\pi_{\bar{a}}(\tau,\vec{\sigma}), \qquad \tilde{\Pi}^u(\tau,\vec{\sigma}) = [e^{-q_u}\rho_u](\tau,\vec{\sigma}),\\
{}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}) &= \sum_s \int d^3\sigma_1 \mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]\cdot\\
&\cdot(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}})(\tau,\vec{\sigma}_1)\Big[\frac{1}{3}\rho + \sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}s}\pi_{\bar{b}}\Big](\tau,\vec{\sigma}_1),\\
\mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}] &= \delta^r_{(a)}\delta^r_s\delta^3(\vec{\sigma},\vec{\sigma}_1) + \mathcal{T}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}],\\
\mathcal{T}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|q,r_{\bar{a}}] &= \frac{1}{2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}r}r_{\bar{c}}(\tau,\vec{\sigma})}\Big[\sum_{w\not= s}\delta_{(k)w}e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}(\gamma_{\bar{c}w}-\gamma_{\bar{c}s})r_{\bar{c}}(\tau,\vec{\sigma}_1)}\cdot\\
&\cdot\Big(\frac{\partial q(\tau,\vec{\sigma}_1)}{\partial\sigma_1^w} + \frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}s}\frac{\partial r_{\bar{c}}(\tau,\vec{\sigma}_1)}{\partial\sigma_1^w}\Big)e^{-q(\tau,\vec{\sigma})}\delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) +\\
&+\delta_{(k)s}\frac{\partial}{\partial\sigma_1^s}e^{-q(\tau,\vec{\sigma})}\delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big],\\
&e^{-q(\tau,\vec{\sigma})}\delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) =\\
&= e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}r}r_{\bar{c}}(\tau,\vec{\sigma})}d^r_{\gamma_{PP_1}}\Big(P_{\gamma_{PP_1}}\, e^{\int_{\vec{\sigma}_1}^{\vec{\sigma}} d\sigma_2^w\, {}^3\hat{\omega}_{w(c)}(\tau,\vec{\sigma}_2)\hat{R}^{(c)}}\Big)_{(a)(k)} +
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_u \delta_{(a)u} e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u} r_{\bar{c}}(\tau,\vec{\sigma})} d^u_{\gamma_{PP_1}}(\vec{\sigma},\vec{\sigma}_1)\delta^r_{(b)}\Big(P_{\gamma_{PP_1}}\, e^{\int_{\vec{\sigma}_1}^{\vec{\sigma}} d\sigma_2^w\, {}^3\hat{\omega}_{w(c)}(\tau,\vec{\sigma}_2)\hat{R}^{(c)}}\Big)_{(b)(k)},\\
&{}^3\hat{K}_{rs} = \frac{\epsilon}{4k} e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}(\gamma_{\bar{c}r}+\gamma_{\bar{c}s})r_{\bar{c}}} \sum_u \Big(\delta_{ru}\delta_{(a)s} + \delta_{su}\delta_{(a)r} - \delta_{rs}\delta_{(a)u}\Big) e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u} r_{\bar{c}}}\, {}^3\hat{\tilde{\pi}}^u_{(a)},\\
&{}^3\hat{K} = -\frac{\epsilon}{4k} e^{-2q} \sum_u \delta_{(a)u} e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u} r_{\bar{c}}}\, {}^3\hat{\tilde{\pi}}^u_{(a)},\\
&{}^3\hat{\tilde{\Pi}}^{rs} = \frac{1}{4} e^{-q}\Big[e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r} r_{\bar{a}}}\, \delta^r_{(a)}\, {}^3\hat{\tilde{\pi}}^s_{(a)} + e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s} r_{\bar{a}}}\, \delta^s_{(a)}\, {}^3\hat{\tilde{\pi}}^r_{(a)}\Big],\\
&{}^3\hat{\tilde{\Pi}} = -2\epsilon k e^{3q}\, {}^3\hat{K} = \frac{1}{2}\sum_r e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r} r_{\bar{a}}}\, \delta^r_{(a)}\, {}^3\hat{\tilde{\pi}}^r_{(a)},
\end{aligned}
\tag{99}
$$

where $\gamma_{\bar{a}u}$ are numerical constants satisfying [5]

$$
\sum_u \gamma_{\bar{a}u} = 0, \qquad \sum_u \gamma_{\bar{a}u}\gamma_{\bar{b}u} = \delta_{\bar{a}\bar{b}}, \qquad \sum_{\bar{a}} \gamma_{\bar{a}u}\gamma_{\bar{a}v} = \delta_{uv} - \frac{1}{3}. \tag{100}
$$

In terms of these variables we have (we reintroduce $n_r \neq 0$ to take into account more general situations)

$$
\begin{aligned}
&{}^3\hat{g}_{rs} = e^{2q}\begin{pmatrix} e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}1} r_{\bar{a}}} & 0 & 0\\ 0 & e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}2} r_{\bar{a}}} & 0\\ 0 & 0 & e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}3} r_{\bar{a}}}\end{pmatrix} = e^{2q}\; {}^3\hat{g}^{diag}_{rs},\\
&\hat{\gamma} = {}^3\hat{g} = {}^3\hat{e}^2 = e^{6q}, \qquad det\, |\hat{g}^{diag}_{rs}| = 1,\\
&d\hat{s}^2 = \epsilon\Big([N_{(as)} + n]^2 - e^{-2q}\sum_u e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u} r_{\bar{a}}}[N_{(as)u} + n_u]^2\Big)(d\tau)^2 -\\
&\quad -2\epsilon[N_{(as)r} + n_r] d\tau d\sigma^r - \epsilon e^{2q}\sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u} r_{\bar{a}}}(d\sigma^u)^2 =\\
&\quad = \epsilon\Big([N_{(as)} + n]^2 (d\tau)^2 -\\
&\quad -\delta_{uv}[e^q e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u} r_{\bar{a}}} d\sigma^u + e^{-q} e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u} r_{\bar{a}}}(N_{(as)u} + n_u) d\tau]\\
&\quad [e^q e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}v} r_{\bar{a}}} d\sigma^v + e^{-q} e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}v} r_{\bar{a}}}(N_{(as)v} + n_v) d\tau],\\
&q = \frac{1}{6}\, ln\, {}^3\hat{g},\\
&r_{\bar{a}} = \frac{\sqrt{3}}{2}\sum_r \gamma_{\bar{a}r}\, ln\, \frac{{}^3\hat{g}_{rr}}{{}^3\hat{g}}.
\end{aligned}
\tag{101}
$$

The momenta ${}^3\hat{\tilde{\pi}}^r_{(a)}$ and ${}^3\hat{\tilde{\Pi}}^{rs}$ and the mean extrinsic curvature ${}^3\hat{K}$ are linear functions of the new momenta ρ and $r_{\bar{c}}$, but with a coordinate-dependent integral kernel. The variables ρ and $r_{\bar{a}}$ replace ${}^3\hat{K}$ and ${}^3\hat{K}^{rs}_{TT}$ [or ${}^3\hat{\tilde{\Pi}}$ and ${}^3\hat{\tilde{\Pi}}^{rs}_{TT}$] of the conformal approach respectively (see

Appendix C) after the solution of the supermomentum constraints (i.e. after the determination of the gravitomagnetic potential) in the 3-orthogonal gauges. It would be important to find the expression of ρ and $r_{\bar a}$ in terms of ${}^3\hat g_{rs}$ and ${}^3\hat K_{rs}$ [or ${}^3\hat{\tilde\Pi}^{rs}$].

In terms of the variables $q, r_{\bar a}$, we have the following results

$$
\begin{aligned}
&{}^3\hat e_{(a)r} = \delta_{(a)r} e^{q_r} = \delta_{(a)r} e^{q+\frac{1}{\sqrt 3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}\\
&\rightarrow_{r_{\bar a}\rightarrow 0}\, \delta_{(a)r}e^q \rightarrow_{q\rightarrow 0}\, \delta_{(a)r},\quad \rightarrow_{q\rightarrow 0}\, \delta_{(a)r} e^{\frac{1}{\sqrt 3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}},\\
&{}^3\hat e^r_{(a)} = \delta^r_{(a)} e^{-q_r} = \delta^r_{(a)} e^{-q-\frac{1}{\sqrt 3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}\\
&\rightarrow_{r_{\bar a}\rightarrow 0}\, \delta^r_{(a)}e^{-q} \rightarrow_{q\rightarrow 0}\, \delta^r_{(a)},\quad \rightarrow_{q\rightarrow 0}\, \delta^r_{(a)} e^{-\frac{1}{\sqrt 3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}},\\
&\\
&{}^3\hat g_{rs} = \delta_{rs} e^{2q_r} = \delta_{rs} e^{2q+\frac{2}{\sqrt 3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}\\
&\rightarrow_{r_{\bar a}\rightarrow 0}\, \delta_{rs}e^{2q} \rightarrow_{q\rightarrow 0}\, \delta_{rs},\quad \rightarrow_{q\rightarrow 0}\, \delta_{rs} e^{\frac{2}{\sqrt 3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}},\\
&{}^3\hat g^{rs} = \delta^{rs} e^{-2q_r} = \delta^{rs} e^{-2q-\frac{2}{\sqrt 3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}\\
&\rightarrow_{r_{\bar a}\rightarrow 0}\, \delta^{rs}e^{-2q} \rightarrow_{q\rightarrow 0}\, \delta^{rs},\quad \rightarrow_{q\rightarrow 0}\, \delta^{rs} e^{-\frac{2}{\sqrt 3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}},\\
&\\
&{}^3\hat e = \sqrt{\hat\gamma} = e^{\sum_r q_r} = e^{3q} \rightarrow_{q\rightarrow 0} 1,\\
&\\
&{}^3\hat\Gamma^r_{uv} = -\delta_{uv}\sum_s \delta^r_s e^{2(q_u-q_s)}\partial_s q_u + \delta^r_u\partial_v q_u + \delta^r_v \partial_u q_v =\\
&= -\delta_{uv}\sum_s \delta^r_s e^{\frac{2}{\sqrt 3}\sum_{\bar a}(\gamma_{\bar au}-\gamma_{\bar as})r_{\bar a}}\Big[\partial_s q + \frac{1}{\sqrt 3}\sum_{\bar b}\gamma_{\bar bu}\partial_s r_{\bar b}\Big] +\\
&+\delta^r_u\Big[\partial_v q + \frac{1}{\sqrt 3}\sum_{\bar a}\gamma_{\bar au}\partial_v r_{\bar a}\Big] + \delta^r_v\Big[\partial_u q + \frac{1}{\sqrt 3}\sum_{\bar a}\gamma_{\bar av}\partial_u r_{\bar a}\Big]\\
&\rightarrow_{r_{\bar a}\rightarrow 0}\, -\delta_{uv}\sum_s \delta^r_s\partial_s q + \delta^r_u\partial_v q + \delta^r_v\partial_u q \rightarrow_{q\rightarrow 0} 0,\\
&\rightarrow_{q\rightarrow 0}\, \frac{1}{\sqrt 3}\Big(-\delta_{uv}\sum_s \delta^r_s e^{\frac{2}{\sqrt 3}\sum_{\bar a}(\gamma_{\bar au}-\gamma_{\bar as})r_{\bar a}}\sum_{\bar b}\gamma_{\bar bu}\partial_s r_{\bar b} + \sum_{\bar a}\Big[\delta^r_u\gamma_{\bar au}\partial_v r_{\bar a} + \delta^r_v\gamma_{\bar av}\partial_u r_{\bar a}\Big]\Big),\\
&\sum_u {}^3\hat\Gamma^u_{uv} = \partial_v \sum_u q_u = 3\partial_v q,\\
&\\
&{}^3\hat\omega_{r(a)} = \epsilon_{(a)(b)(c)}\delta_{(b)r}\delta_{(c)u} e^{q_r-q_u}\partial_u q_r =\\
&= \epsilon_{(a)(b)(c)}\delta_{(b)r}\delta_{(c)u} e^{\frac{1}{\sqrt 3}\sum_{\bar a}(\gamma_{\bar ar}-\gamma_{\bar au})r_{\bar a}}\Big[\partial_u q + \frac{1}{\sqrt 3}\sum_{\bar b}\gamma_{\bar br}\partial_u r_{\bar b}\Big]\\
&\rightarrow_{r_{\bar a}\rightarrow 0}\, \epsilon_{(a)(b)(c)}\delta_{(b)r}\delta_{(c)u}\partial_u q \rightarrow_{q\rightarrow 0} 0,\\
&\rightarrow_{q\rightarrow 0}\, \frac{1}{\sqrt 3}\epsilon_{(a)(b)(c)}\sum_u \delta_{(b)r}\delta_{(c)u} e^{\frac{1}{\sqrt 3}\sum_{\bar a}(\gamma_{\bar ar}-\gamma_{\bar au})r_{\bar a}}\sum_{\bar b}\gamma_{\bar br}\partial_u r_{\bar b}.
\end{aligned}
\tag{102}
$$

See Appendix D for the expression of other 3-tensors and Appendix E for the corresponding expression of 4-tensors.

Since the proper gauge transformations go to the identity at spatial infinity, Eqs.(98), (99), (95), (94) and (6) imply the following boundary conditions

$$
\begin{aligned}
&{}^3\hat{e}_{(a)r}(\tau,\vec{\sigma}) = \delta_{(a)r} Q_r(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \delta_{(a)r} + \frac{{}^3\hat{w}_{(as)(a)r}(\tau)}{r} + O(r^{-2}),\\
&Q_r(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} 1 + \frac{Q_{(as)r}(\tau)}{r} + O(r^{-2}), \quad {}^3\hat{w}_{(as)(a)r}(\tau) = \delta_{(a)r} Q_{(as)r}(\tau),\\
&q(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \frac{1}{3r}\sum_u Q_{(as)u}(\tau) + O(r^{-2}),\\
&r_{\bar{a}}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \frac{\sqrt{3}}{r}\sum_u \gamma_{\bar{a}u} Q_{(as)u}(\tau) + O(r^{-2}),\\
&{}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \frac{{}^3\hat{\tilde{p}}^r_{(as)(a)}(\tau)}{r^2} + O(r^{-3}),\\
&\tilde{\Pi}^r(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \frac{\tilde{\Pi}^r_{(as)}(\tau)}{r^2} + O(r^{-3}),\\
&\rho(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \frac{1}{r^2}\sum_u \gamma_{\bar{a}u}\tilde{\Pi}^u_{(as)}(\tau) + O(r^{-3}),\\
&\pi_{\bar{a}}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \frac{1}{\sqrt{3}\, r^2}\sum_u \gamma_{\bar{a}u}\tilde{\Pi}^u_{(as)}(\tau) + O(r^{-3}),\\
&{}^3\hat{\omega}_{r(a)}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \frac{{}^3\hat{\omega}_{(as)r(a)}(\tau)}{r^2} + O(r^{-3}).
\end{aligned}
\tag{103}
$$

By using Appendix D, we find that ${}^3\hat{R}_{rsuv}(\tau,\vec{\sigma})$ and ${}^3\hat{\Omega}_{rs(a)}(\tau,\vec{\sigma})$ go as $O(r^{-3})$ for $r\rightarrow\infty$.

The superhamiltonian constraint takes the following final reduced form

$$
\begin{aligned}
\hat{\mathcal{H}}_R(\tau,\vec{\sigma}) &= -\epsilon\frac{1}{8k}e^{-\sum_u q_u(\tau,\vec{\sigma})}\Big(\ [2\sum_u \rho_u^2 - (\sum_u \rho_u)^2](\tau,\vec{\sigma}) +\\
&+ 2\sum_{r,s} e^{q_s(\tau,\vec{\sigma})}\rho_r(\tau,\vec{\sigma})(2\delta_{rs}-1)\delta_{(a)s}\int d^3\sigma_1\, \mathcal{T}^s_{(a)v}(\vec{\sigma},\vec{\sigma}_1;\tau|e^{q_t}]\, e^{-q_v(\tau,\vec{\sigma}_1)}\rho_v(\tau,\vec{\sigma}_1) +\\
&+ \sum_{r,s} e^{q_r(\tau,\vec{\sigma})+q_s(\tau,\vec{\sigma})}\int d^3\sigma_1 d^3\sigma_2 \sum_{uv} e^{-q_u(\tau,\vec{\sigma}_1)-q_v(\tau,\vec{\sigma}_2)}\,\rho_u(\tau,\vec{\sigma}_1)\mathcal{T}^r_{(b)u}(\vec{\sigma},\vec{\sigma}_1;\tau|e^{q_t}]\\
&\cdot\ \Big(\delta_{rs}\delta_{(a)(b)} + \delta_{(a)r}\delta_{(b)s} - \delta_{(a)s}\delta_{(b)r}\Big)\mathcal{T}^s_{(a)v}(\vec{\sigma},\vec{\sigma}_2;\tau|e^{q_t}]\,\rho_v(\tau,\vec{\sigma}_2)\ \Big) +\\
&+ \epsilon k\left[e^{\sum_u q_u - q_r - q_s}\right](\tau,\vec{\sigma})\epsilon_{(a)(b)(c)}\delta_{(a)r}\delta_{(b)s}\ {}^3\hat{\Omega}_{rs(c)}[e^{q_t}](\tau,\vec{\sigma}) =\\
&= -\epsilon\frac{e^{-q(\tau,\vec{\sigma})}}{8k}\Big[(e^{-2q}[6\sum_{\bar{a}}\pi^2_{\bar{a}} - \frac{1}{3}\rho^2])(\tau,\vec{\sigma}) +\\
&+ 2(e^{-q}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}[2\sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}u}\pi_{\bar{b}} - \frac{1}{3}\rho])(\tau,\vec{\sigma})\ \times\\
&\quad \int d^3\sigma_1 \sum_r \delta^u_{(a)}\mathcal{T}^r_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}]\Big)(\tau,\vec{\sigma}_1) +\\
&+ \int d^3\sigma_1 d^3\sigma_2\Big(\sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u} + r_{\bar{a}}(\tau,\vec{\sigma})}\ \times\\
&\quad \sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}]\Big)(\tau,\vec{\sigma}_1)\ \times
\end{aligned}
$$

$$\sum_s \mathcal{T}^u_{(a)s}(\vec{\sigma},\vec{\sigma}_2,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}]\Big)(\tau,\vec{\sigma}_2)+$$
$$+\sum_{uv} e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}u}+\gamma_{\bar{a}v})r_{\bar{a}}(\tau,\vec{\sigma})}(\delta^u_{(b)}\delta^v_{(a)}-\delta^u_{(a)}\delta^v_{(b)})\times$$
$$\sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}]\Big)(\tau,\vec{\sigma}_1)$$
$$\sum_s \mathcal{T}^v_{(b)s}(\vec{\sigma},\vec{\sigma}_2,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}]\Big)(\tau,\vec{\sigma}_2)\,\Big)\Big]+$$
$$+\epsilon k\sum_{r,s}\Big[e^{q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}+\gamma_{\bar{a}s})r_{\bar{a}}}\Big](\tau,\vec{\sigma})\epsilon_{(a)(b)(c)}\delta_{(a)r}\delta_{(b)s}\,{}^3\hat{\Omega}_{rs(c)}[q,r_{\bar{c}}](\tau,\vec{\sigma})\approx 0. \qquad (104)$$

The last line is equal to $\epsilon k e^{3q}\,{}^3\hat{R}[q,r_{\bar{a}}]$.

The conformal factor $q(\tau,\vec{\sigma})$ of the 3-metric has been interpreted as an "intrinsic internal time" [it is not a scalar and is proportional to Misner's time $\Omega=-\frac{1}{3}\,ln\,\sqrt{\gamma}$ [53] for asymptotically flat spacetimes (see Appendix C): $q=-\frac{1}{2}\Omega$], to be contrasted with York's "extrinsic internal time" $\mathcal{T}=-\frac{4}{3}\epsilon k\,{}^3K=\frac{2}{3\sqrt{\gamma}}{}^3\tilde{\Pi}$ [see Ref. [58] for a review of the known results with York's extrinsic internal time, Ref. [59] for York cosmic time versus proper time and Refs. [60,61] for more general reviews about the problem of time in general relativity].

Let us also remark that if we would have added only the gauge-fixing $\alpha_{(a)}(\tau,\vec{\sigma})\approx 0$ [so that the associated Dirac brackets would coincide with the ADM Poisson brackets for metric gravity as already said], the four variables $\vec{\xi}(\tau,\vec{\sigma})$, $q(\tau,\vec{\sigma})$ [with conjugate momenta $\tilde{\pi}^{\vec{\xi}}_r(\tau,\vec{\sigma})\approx 0$, $\rho(\tau,\vec{\sigma})$] would correspond to the variables used in Ref. [60] to label the points of the spacetime M^4 (assumed compact), following the suggestion of Ref. [62], if $q(\tau,\vec{\sigma})$ is interpreted as a time variable (see Section VI for a different identification of points). However, the example of the foliation of Minkowski spacetime with rectangular coordinates by means of spacelike hyperplanes, shows that both internal intrinsic [$q(\tau,\vec{\sigma})$] and extrinsic [${}^3K(\tau,\vec{\sigma})$] times cannot be used as time labels to identify the leaves: i) ${}^3K=0$ on every leaf; ii) $q=0$ on every leaf. Therefore, we shall not accept $q(\tau,\vec{\sigma})$ as a time variable for M^4: the problem of time in the Hamiltonian formulation will be discussed in Ref. [6] (see also Section VI). A related problem (equivalent to the transition from a Cauchy problem to a Dirichlet one and requiring a definition of which time parameter has to be used) is the validity of the "full sandwich conjecture" [62,63] [given two nearby 3-metrics on Cauchy surfaces Σ_{τ_1} and Σ_{τ_2}, there is a unique spacetime M^4, satisfying Einstein's equations, with these 3-metrics on those Cauchy surfaces] and of the "thin sandwich conjecture" [given ${}^3g$ and $\partial_\tau\,{}^3g$ on Σ_τ, there is a unique spacetime M^4 with these initial data satisfying Einstein's equations]: see Ref. [64] for the non validity of the "full" case and for the restricted validity (and its connection with constraint theory) of the "thin" case.

See Appendix C for some notions on mean extrinsic curvature slices, for the TT-decomposition, for more comments about internal intrinsic and extrinsic times and for a review of the Lichnerowicz-York conformal approach to the reduction of metric gravity. In this approach the superhamiltonian constraint (namely the elliptic Lichnerowicz equation) is solved in the variable $\phi(\tau,\vec{\sigma})=e^{\frac{1}{2}q(\tau,\vec{\sigma})}$, namely in the conformal factor $q(\tau,\vec{\sigma})$ rather than in its conjugate momentum $\rho(\tau,\vec{\sigma})$. In the conformal approach one uses York's variables [65], because most of the work on the Cauchy problem for Einstein's equations in metric gravity [see the reviews [66,58] with their rich bibliography and Ref. [67], where it is shown (see the end of Appendix C and Eq.(C7) for the notations) that if one extracts the transverse traceless part ${}^3\tilde{\Pi}^{rs}_{TT}$ of ${}^3\tilde{\Pi}^{rs}_A$, one may define a local canonical basis with variables $\mathcal{T}$, $\mathcal{P}_{\mathcal{T}}$, ${}^3\sigma_{rs}$, ${}^3\tilde{\Pi}^{rs}_{TT}$: it is called the York map] is done by using spacelike hypersurfaces Σ of constant mean extrinsic curvature in the compact case [see Refs. [66,68,69]] and with the maximal slicing condition $\mathcal{T}(\tau,\vec{\sigma}) = 0$ (it may be extended to non constant $\mathcal{T}$) in the asymptotically free case [see also Ref. [70] for recent work in the compact case with non constant $\mathcal{T}$ and Ref. [71] for solutions of Einstein's equations in presence of matter which do not admit constant mean extrinsic curvature slices]. Let us remark that in Minkowski spacetime ${}^3K(\tau,\vec{\sigma}) = 0$ are the hyperplanes, while ${}^3K(\tau,\vec{\sigma}) = const.$ are the mass hyperboloids, corresponding to the instant and point form of the dynamics according to Dirac [72] respectively [see Refs. [73] for other types of foliations]. It would be extremely important to have some ideas how to find explicitly the canonical transformation from our canonical basis in the 3-orthogonal gauges to the canonical basis whose existence is assured by the York map [67].

Instead in Ref. [58] in the case of compact spacetimes the superhamiltonian constraint is interpreted as a "time-dependent Hamiltonian" for general relativity in the intrinsic internal time q .

We shall see in Ref. [6], that in asymptotically flat (at spatial infinity) spacetimes the canonically reduced superhamiltonian constraint $\hat{\mathcal{H}}_R(\tau,\vec{\sigma}) \approx 0$ in 3-orthogonal coordinates has to be interpreted (like in the conformal approach) as an integrodifferential equation, the reduced Lichnerowicz equation, for the conformal factor $\phi(\tau,\vec{\sigma}) = e^{\frac{1}{2}q(\tau,\vec{\sigma})}$ whose solution gives it as a functional of the canonical variables $r_{\bar{a}}(\tau,\vec{\sigma})$, $\pi_{\bar{a}}(\tau,\vec{\sigma})$, and of the last gauge variable: the momentum $\pi_\phi(\tau,\vec{\sigma}) = 2\phi^{-1}(\tau\vec{\sigma})\rho(\tau,\vec{\sigma})$ conjugate to the conformal factor. The evolution in τ (the time parameter labelling the leaves Σ_τ of the foliation associated with the 3+1 splitting of M^4) will be shown to be generated by the ADM energy [absent in closed spacetimes]. The solution $\phi = e^{q/2} \approx e^{F[r_{\bar{a}},\pi_{\bar{a}},\rho]}$ of the reduced Lichnerowicz equation determines an equivalence class of 3-geometries (or conformal 3-geometry) parametrized by the gauge variable $\rho(\tau,\vec{\sigma})$ [conformal gauge orbit]; the natural representative of an equivalence class is obtained with the gauge-fixing $\rho(\tau,\vec{\sigma}) \approx 0$: ${}^3g_{rs} = e^{4F[r_{\bar{a}},\pi_{\bar{a}},0]}\, {}^3\hat{g}^{diag}_{rs}[r_{\bar{a}},\pi_{\bar{a}}]$.

The functions $r_{\bar{a}}(\tau,\vec{\sigma})$, $\bar{a} = 1,2$, give a parametrization of the Hamiltonian physical degrees of freedom of the gravitational field and of the space of conformal 3-geometries [the quotient of superspace by the group $Weyl\,\Sigma_\tau$, if by varying ρ the solution $\phi = e^{q/2} \approx e^{F[r_{\bar{a}},\pi_{\bar{a}},\rho]}$ of the Lichnerowicz equation spans all the Weyl rescalings]: it turns out that a point (a 3-geometry) in this space, i.e. a ${}^3\hat{g}^{diag}_{rs}$ [it is simultaneously the York [65] reduced metric and the Misner's one [53] in 3-orthogonal coordinates; see the end of Section VI], is a class of conformally related 3-metrics (conformal gauge orbit).

When we add the gauge-fixing $\rho(\tau,\vec{\sigma}) \approx 0$ to the superhamiltonian constraint and go to Dirac brackets eliminating the conjugate variables $q(\tau,\vec{\sigma})$, $\rho(\tau,\vec{\sigma})$, the functions $r_{\bar{a}}(\tau,\vec{\sigma})$ and $\pi_{\bar{a}}(\tau,\vec{\sigma})$ become the physical canonical variables for the gravitational field in this special 3-orthogonal gauge; this does not happens with the gauge-fixing ${}^3K(\tau,\vec{\sigma}) \approx 0\,(or\, const.)$. The ADM energy, which, in this gauge, depends only on $r_{\bar{a}}$, $\pi_{\bar{a}}$ is the Hamiltonian generating the τ-evolution of the physical (non covariant) gravitational field degrees of freedom [this corresponds to the two dynamical equations contained in the 10 Einstein equations in this gauge]. In Ref. [6] there will be a more complete discussion of these points.

Since there are statements [see Ref. [74]; Ref. [75] contains a recent review with a rich bibliography] on the existence and unicity of solutions of the 5 equations of ADM metric gravity (the Lichnerowicz equation or superhamiltonian constraint, the 3 supermomentum constraints and the gauge fixing (maximal slicing condition) ${}^3K(\tau,\vec{\sigma}) = 0$) and since our approach to tetrad gravity contains metric gravity, it is reasonable that this demonstration can be extended to the reduced Lichnerowicz equation [obtained by putting into it a solution of the supermomentum constraints possible only in tetrad gravity since it uses the Green function (42)] with the gauge fixing ${}^3K(\tau,\vec{\sigma}) = 0$ replaced with $\rho(\tau,\vec{\sigma}) = 0$.

Let us remark that Minkowski spacetime in Cartesian coordinates is a solution of Einstein equations, which in the 3-orthogonal gauges corresponds to $q = \rho = r_{\bar{a}} = \pi_{\bar{a}} = 0$ [$\phi = 1$] and $n = n_r = N_{(as)r} = 0$, $N_{(as)} = \epsilon$ [for $q = \rho = r_{\bar{a}} = 0$ Eq.(88) implies ${}^3\hat{\tilde{\pi}}^r_{(a)}$ proportional to $\pi_{\bar{a}}$; the condition $\Sigma_\tau = R^3$ implies ${}^3K_{rs} = 0$ and then $\pi_{\bar{a}} = 0$].

Therefore, it is consistent with Einstein equations to add by hand the two pairs of second class constraints $r_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$, $\pi_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$, to the Dirac Hamiltonian with arbitrary multipliers,

$$H^{'}_{(D)ADM,R} = H_{(D)ADM,R} + \int d^3\sigma [\sum_{\bar{a}} (\mu_{\bar{a}} r_{\bar{a}} + \nu_{\bar{a}} \pi_{\bar{a}})](\tau,\vec{\sigma}).$$

The time constancy of these second class constraints determines the multipliers

$$\begin{aligned}
&\partial_\tau r_{\bar{a}}(\tau,\vec{\sigma}) \stackrel{\circ}{=} \nu_{\bar{a}}(\tau,\vec{\sigma}) + \int d^3\sigma_1 n(\tau,\vec{\sigma}_1) \{ r_{\bar{a}}(\tau,\vec{\sigma}), \hat{\mathcal{H}}_R(\tau,\vec{\sigma}_1)\}^* \approx 0,\\
&\partial_\tau \pi_{\bar{a}}(\tau,\vec{\sigma}) \stackrel{\circ}{=} -\mu_{\bar{a}}(\tau,\vec{\sigma}) + \int d^3\sigma_1 n(\tau,\vec{\sigma}_1) \{ \pi_{\bar{a}}(\tau,\vec{\sigma}), \hat{\mathcal{H}}_R(\tau,\vec{\sigma}_1)\}^* \approx 0.
\end{aligned}$$

By going to new Dirac brackets, we remain with the only conjugate pair $q(\tau,\vec{\sigma})$, $\rho(\tau,\vec{\sigma})$, constrained by the first class constraint $\hat{\mathcal{H}}_R(\tau,\vec{\sigma})|_{r_{\bar{a}}=\pi_{\bar{a}}=0} \approx 0$. In this way we get the description of a family of gauge equivalent spacetimes M^4 without gravitational field (see Ref. [6]), which could be called a "void spacetime", with 3-orthogonal coordinates for Σ_τ. They turn out to be "3-conformally flat" because ${}^3\hat{g}_{rs} = e^q\, \delta_{rs}$. Now, the last of Eqs.(99) [with $r_{\bar{a}} = \pi_{\bar{a}} = 0$] is an integral equation to get ρ in terms of ${}^3\hat{\tilde{\Pi}}$ [or ${}^3\hat{K}$] and $q = \frac{1}{6}\, ln\, {}^3\hat{g}$.

If we add the extra gauge-fixing $\rho(\tau,\vec{\sigma}) \approx 0$, we get the 3-Euclidean metric δ_{rs} on Σ_τ, since the superhamiltonian constraint has $q(\tau,\vec{\sigma}) \approx 0$ [$\phi(\tau,\vec{\sigma}) \approx 1$] as a solution in absence of matter. The time constancy of $\rho(\tau,\vec{\sigma}) \approx 0$ implies $n(\tau,\vec{\sigma}) \approx 0$. Indeed, for the reduction to Minkowski spacetime, besides the solution $q(\tau,\vec{\sigma}) \approx 0$ of the superhamiltonian constraint [vanishing of the so called internal intrinsic (many-fingered) time [76]], we also need the gauge-fixings $N_{(as)}(\tau,\vec{\sigma}) \approx \epsilon$, $N_{(as)r}(\tau,\vec{\sigma}) \approx 0$, $n_r = 0$. Many members of the equivalence class of void spacetimes represent flat Minkowski spacetimes in the most arbitrary coordinates compatible with Einstein theory with the associated inertial effects as in Newtonian gravity in noninertial Galilean frames. Therefore, they seem to represent the most general "pure acceleration effects without gravitational field (i.e. without tidal effects) but with a control on the boundary conditions" compatible with Einstein's general relativity for globally hyperbolic, asymptotically flat at spatial infinity spacetimes [see also the discussion on general covariance and on the various formulations of the equivalence principle (homogeneous gravitational fields = absence of tidal effects) in Norton's papers [77]].

Void spacetimes can be characterized by adding to the ADM action the Cotton-York 3-conformal tensor with Lagrange multiplier [see Appendix D and Eq.(D2)], but this will be studied elsewhere.

See the future papers [6,30] for the use of this reduced symplectic structure for a solution of the deparametrization problem in general relativity in presence of matter.

VI. CONCLUSIONS AND INTERPRETATIONAL PROBLEMS: DIRAC'S OBSERVABLES VERSUS GENERAL COVARIANCE AND BERGMANN'S SPACETIME COORDINATES.

In this second paper dealing with a new formulation of tetrad gravity on globally hyperbolic, asymptotically flat at spatial infinity, spacetimes with Cauchy 3-surfaces Σ_τ diffeomorphic to R^3 (so that they admit global coordinate systems), we analyzed the Hamiltonian group of gauge transformations whose generators are the 14 first class constraints of the model. After introducing a new parametrization of the lapse and shift functions, suited for spacetimes asymptotically flat at spatial infinity, we studied in detail the subgroup of gauge transformations associated with rotations and pseudodiffeomorphisms of the cotriads, namely the automorphism group of the coframe SO(3) principal bundle. We pointed out the necessity of using weighted Sobolev spaces to avoid the presence of stability subgroups of gauge transformations generating the cone over cone structure of singularities on the constraint manifold and creating an obstruction to the canonical reduction [Gribov ambiguity of the spin connection and isometries of the 3-metric]. This description will be valid, in a variational sense, for a finite interval $\triangle\tau$, after which conjugate points for the 3-geometry of Σ_τ and/or 4-dimensional singularities will develop due to Einstein equations.

Then we defined and solved the multitemporal equations for cotriads and their conjugate momenta on Σ_τ associated with rotations and spatial pseudodiffeomorphisms. This required a proposal for the parametrization of the group manifold of these gauge transformations. Also the corresponding six first class constraints have been solved and Abelianized. The final outcome was the explicit dependence of cotriads and their momenta on the three rotation angles and on the three pseudodiffeomorphisms parameters. The Dirac observables with respect to these gauge transformations are reduced cotriads depending only on three arbitrary functions [the reduced momenta also depend on the momenta conjugate to these functions]. We have shown that the choice of the coordinate system on Σ_τ is equivalent to the choice of how to parametrize the reduced cotriads in terms of the three arbitrary functions, and this also gives a parametrization of the superspace of 3-geometries ($Riem\,\Sigma_\tau/Diff\,\Sigma_\tau$).

By choosing a parametrization corresponding to global 3-orthogonal coordinate systems on $\Sigma_\tau \approx R^3$, we were able to perform a global (at least at a heuristic level) quasi-Shanmugadhasan canonical transformation to a canonical basis in which 13 first class constraints are Abelianized. Next we defined the Dirac brackets corresponding to 3-orthogonal gauges (choice of 3-orthogonal coordinates and of the origin of angles; there is a residual gauge freedom corresponding to rotating 3-orthogonal gauges; the choice of a congruence of timelike observers now depends only on the choice of the 3 boost parameters $\varphi_{(a)}$), we made a further canonical transformation to more transparent canonical variables and reexpressed all 3- and 4-tensors in this final basis. Besides lapse and shift functions, the final configuration variables for the superspace of 3-geometries are the conformal factor of the 3-metric $\phi = e^{q/2} = \gamma^{1/12}$ and two functions $r_{\bar a}$ (the genuine degrees of freedom of the gravitational field) parametrizing the diagonal elements of the 3-metric. Moreover, there are the two momenta $\pi_{\bar a}$ of the gravitational field and the momentum ρ [$\pi_\phi = 2\phi^{-1}\rho$] conjugate to the conformal factor q [ϕ]. The momentum ρ, containing a nonlocal information on the extrinsic curvature of Σ_τ, and not ${}^3K$ (which depends non locally upon ρ) is the last gauge variable of tetrad gravity. The only left first class constraint is the reduced superhamiltonian one, which becomes an integrodifferential equation for the conformal factor [in metric gravity it would correspond to the Lichnerowicz equation after having put into it the solution of the supermomentum constraints] as it will be justified in Ref. [6]. A comparison has been made with the conformal approach of Lichnerowicz and York.

In future papers [6,30] there will be the study of the superhamiltonian constraint (with the refusal of its quantum version, the Wheeler-DeWitt equation, as an evolution equation), of the asymptotic Poincaré charges, of the ADM energy as the physical Hamiltonian (and of the related problem of time), of the deparametrization of tetrad gravity in presence of matter (scalar particles) . In this way, we will see that the 3-orthogonal gauges are the equivalent of the Coulomb gauge in classical electrodynamics (like the harmonic gauge is the equivalent of the Lorentz gauge): this will allow to show explicitly the action-at-a-distance (Newton-like and gravitomagnetic) potentials among particles hidden in tetrad gravity (like the instantaneous Coulomb potential is hidden in the electromagnetic gauge potential). Spinning particles will be needed to study precessional effects from gravitomagnetism. Also a reformulation of the canonical reduction done in this paper in local normal coordinates on Σ_τ will be needed as a first step towards the study of normal coordinates in M^4, necessary to define local nonrotating inertial observers and to study the geodetic deviation equation.

Our approach breaks the general covariance of general relativity completely by going to the special 3-orthogonal gauge with $\rho(\tau, \vec{\sigma}) \approx 0$. But this is done in a way naturally associated with presymplectic theories (i.e. theories with first class constraints like all formulations of general relativity and the standard model of elementary particles with or without supersymmetry): the global Shanmugadhasan canonical transformations (when they exist; for instance they do not exist when the configuration space is compact like in closed spacetimes) correspond to privileged Darboux charts for presymplectic manifolds. Therefore, the gauges identified by these canonical transformations should have a special (till now unexplored) role also in generally covariant theories, in which traditionally one looks for observables invariant under spacetime diffeomorphisms (but no complete basis is known for them in general relativity) and not for (not generally covariant) Dirac observables. While in electromagnetism and in Yang-Mills theories the physical interpretation of Dirac observables is clear, in generally covariant theories there is a lot of interpretational problems and ambiguities.

Therefore, let us finish with some considerations on interpretational problems, whose relevance has been clearly pointed out in Ref. [78].

First of all, let us interpret metric and tetrad gravity according to Dirac-Bergmann theory of constraints (the presymplectic approach). Given a mathematical noncompact, topologically trivial, manifold M^4 with a maximal C^∞-atlas A, its diffeomorphisms in $Diff\, M^4$ are interpreted in passive sense (pseudodiffeomorphisms): chosen a reference atlas (contained in A) of M^4, each pseudodiffeomorphism identifies another possible atlas contained in A. The pseudodiffeomorphisms are assumed to tend to the identity at spatial infinity in a way which will be discussed in the next paper [6]. Then we add an arbitrary C^∞ metric structure on M^4, we assume that $(M^4, {}^4g)$ is globally hyperbolic and asymptotically flat at spatial infinity and we arrive at a family of Lorentzian spacetimes $(M^4, {}^4g)$ over M^4. On $(M^4, {}^4g)$ one usually defines [63,79] the standards of length and time, by using some material bodies, with the help of mathematical structures like the line element ds^2, timelike geodesics (trajectories of test particles) and null geodesics (trajectories of photons), without any reference to Einstein's equations [see the conformal, projective, affine and metric structures hidden in $(M^4, {}^4g)$ according to Ref. [80], which replace at the mathematical level the "material reference frame" concept [81–83] with its 'test' objects]; only the equivalence principle (statement about test particles in an external given gravitational field) is used to emphasize the relevance of geodesics. Let $\tilde{Diff}\, M^4$ be the extension of $Diff\, M^4$ to the space of tensors over M^4. Since the Hilbert action of metric gravity is invariant under the combined action of $Diff\, M^4$ and $\tilde{Diff}\, M^4$, one says that the relevant object in gravity is the set of all 4-geometries over M^4 [$(M^4, {}^4g)$ modulo $Diff\, M^4$ or $Riem\, M^4/Diff\, M^4$] and that the relevant quantities (generally covariant observables) associated with it are the invariants under diffeomorphisms like the curvature scalars. From the point of view of dynamics, one has to select those special 4-geometries whose representatives $(M^4, {}^4g)$ satisfy Einstein's equations, which are invariant in form under diffeomorphisms (general covariance). The variation of a solution ${}^4g_{\mu\nu}(x)$ of Einstein's equations under infinitesimal spacetime diffeomorphisms, namely $\mathcal{L}_{\xi^\rho\partial_\rho}\, {}^4g_{\mu\nu}(x)$, satisfies the Jacobi equations associated with Einstein's equations or linearized Einstein equations [see Refs. [84–86]; with our assumptions we are in the non-compact case (like Ref. [86]) without Killing vectors: in this case it is known that near Minkowski spacetime the Einstein empty space equations are linearization stable]: therefore these Noether (gauge) symmetries of the Hilbert action are also dynamical symmetries of Einstein equations.

One can say that a "kinematical gravitational field" is a 4-geometry (an element of $Riem\, M^4/Diff\, M^4$), namely an equivalence class of 4-metrics modulo $Diff\, M^4$, and that an "Einstein or dynamical gravitational field" (or Einstein 4-geometry or equivalence class of Einstein spacetimes) is a kinematical gravitational field which satisfies Einstein's equations.

However, the fact that the ten Einstein equations are not a hyperbolic system of differential equations and cannot be put in normal form [this is evident if one starts with the ADM action, because the ADM Lagrangian is singular] is only considered in connection with the initial data problem. Instead, the ADM action (needed as the starting point to define the canonical formalism since it has a well posed variational problem) contains the extra input of a 3+1 splitting of M^4: this allows the identification of the surface term containing the second time derivatives of the 4-metric to be discarded from the Hilbert action. As a consequence the ADM action is quasi-invariant under the pullback of the Hamiltonian group of gauge transformations generated by the first class constraints (as every singular Lagrangian) and this group is not $Diff\, M^4$ plus its extension $\tilde{Diff}\, M^4$, as it will be shown in Ref. [6]. In particular, the ADM action is not invariant under diffeomorphisms in $Diff\, M^4$ skew with respect to the foliation of M^4 associated to the chosen 3+1 splitting, even if the ADM theory is independent from the choice of the 3+1 splitting. However, the ADM action generates the same equations of motion, i.e. Einstein's equations, so that the space of the dynamical symmetries of these equations is the same as in the description base on the Hilbert action: but now not every dynamical symmetry of Einstein's equations is a Noether symmetry of the ADM action.

Regarding the 10 Einstein equations, the Bianchi identities imply that four equations are linearly dependent on the other six ones and their gradients. Moreover, the four combinations of Einstein's equations projectable to phase space (where they become the secondary first class superhamitonian and supermomentum constraints of canonical metric and tetrad gravity) are independent from the accelerations and are only restrictions on the Cauchy data. As a consequence, the Einstein equations have solutions, in which the ten components ${}^4g_{\mu\nu}$ of the 4-metric depend on only two dynamical (but not tensorial) degrees of freedom (defining the physical gravitational field) and on eight undetermined degrees of freedom [more exactly the four components of the 4-metric corresponding to the lapse and shift functions and on the four functions depending on the gradients of the 4-metric (generalized velocities) corresponding, through the first half of Hamilton equations, to the four arbitrary Dirac multipliers in front of the primary constraints (vanishing of the momenta conjugate to lapse and shift functions) in the Dirac Hamiltonian].

This transition from the ten components ${}^4g_{\mu\nu}$ of the tensor ${}^4g$ in some atlas of M^4 to the 2 (deterministic)+8 (undetermined) degrees of freedom breaks general covariance, because these quantities are neither tensors nor invariants under spacetime diffeomorphisms (their functional form is atlas dependent in a way dictated by the 3+1 splittings of M^4 needed for defining the canonical formalism). This is manifest in the canonical approach (we discuss metric gravity, but nothing changes in tetrad gravity except that there are six more undetermined degrees of freedom):
i) choose an atlas for M^4, a 3+1 splitting M^{3+1} of M^4 (with leaves Σ_τ of the foliation assumed diffeomorphic to R^3), go to coordinates adapted to the 3+1 splitting [atlas for M^{3+1} with coordinate charts $(\sigma^A) = (\tau, \vec{\sigma})$, connected to the M^4 atlas by the transition functions $b^\mu_A(\tau, \vec{\sigma})$ of Section II of I] and replace $Diff\, M^4$ with $Diff\, M^{3+1}$ (the diffeomorphisms respecting the 3+1 splitting);
ii) the ten components ${}^4g_{AB}$ of the 4-metric in the adapted coordinates are non covariantly replaced with N, N^r, ${}^3g_{rs}$, whose conjugate momenta are $\tilde{\pi}_N$, $\tilde{\pi}^{\vec{N}}_r$, ${}^3\tilde{\Pi}^{rs}$;
iii) there are four primary $[\tilde{\pi}_N \approx 0,\ \tilde{\pi}^{\vec{N}}_r \approx 0]$ and four secondary $[\tilde{\mathcal{H}} \approx 0,\ \tilde{\mathcal{H}}^r \approx 0]$ first class constraints;
iv) therefore, the twenty canonical variables have to be replaced (with a Shanmugadhasan canonical transformation) with two pairs of genuine physical degrees of freedom (Dirac's observables), with eight gauge variables and with eight abelianized first class constraints;
v) this separation is dictated by the Hamiltonian group $\bar{\mathcal{G}}$ of gauge transformations which has eight generators and is not connected with $\tilde{D}iff\, M^{3+1}$ [except for spatial diffeomorphisms $Diff\, \Sigma_\tau \subset Diff\, M^{3+1}$], which has only four generators and whose invariants are not Dirac observables [the so called time-diffeomorphisms are replaced by the 5 gauge transformations generated by $\tilde{\pi}_N$, $\tilde{\pi}^{\vec{N}}_r$, and the superhamiltonian constraint];
vi) as already said at the end of Section V of I, the eight gauge variables should be fixed by giving only four gauge fixings for the secondary constraints (the same number of conditions needed to fix a diffeomorphisms), because their time constancy determines the four secondary gauge fixings for the primary constraints [and, then, their time constancy determines the Dirac multipliers (four velocity functions not determined by Einstein equations) in front of the primary constraints in the Dirac Hamiltonian].

Since no one has solved the metric gravity secondary constraints till now, it is not clear what is undetermined inside ${}^3g_{rs}$ (see Appendix C for what is known from the conformal approach) and, therefore, which is the physical meaning (with respect to the arbitrary determination of the standards of length and time) of the first four gauge-fixings. Instead, the secondary four gauge-fixings determine the lapse and shift functions, namely they determine how the leaves Σ_τ are packed in the foliation (the gauge nature of the shift functions, i.e. of ${}^4g_{oi}$, is connected with the conventionality of simultaneity [88]). Let us remark that the invariants under spacetime diffeomorphisms are in general not Dirac observables, because they depend on the eight gauge variables not determined by Einstein's equations. Therefore, all the curvature scalars are gauge quantities at least at the kinematical level, as can be deduced from the expression of 4-tensors given in Appendix B of I and in Appendix E.

In this paper we have clarified the situation in the case of tetrad gravity, and, as a consequence, also for metric gravity since we started from the ADM action. The original 32 canonical variables N, $N_{(a)}$, $\varphi_{(a)}$, ${}^3e_{(a)r}$, $\tilde{\pi}_N$, $\tilde{\pi}^{\vec{N}}_{(a)}$, $\tilde{\pi}^{\vec{\varphi}}_{(a)}$, ${}^3\tilde{\pi}^r_{(a)}$ (we disregard the asymptotic part of the lapse and shift functions for this discussion) have been replaced, in the case of 3-orthogonal coordinates $\vec{\sigma}$ on Σ_τ and therefore in the associated coordinates $(\tau, \vec{\sigma})$ of an atlas of M^{3+1}, by the Dirac's observables $r_{\bar{a}}$, $\pi_{\bar{a}}$ [the gravitational field], by 14 first class constraints [13 have been abelianized] and by 14 gauge variables: N, $N_{(a)}$, $\varphi_{(a)}$, $\alpha_{(a)}$, ξ^r, ρ [the momentum conjugate to the conformal factor q of the 3-metric; q is determined by the superhamiltonian constraint or Lichnerowicz equation]. Now we have to add 10 primary gauge-fixings:

i) 6 gauge-fixings, determining $\varphi_{(a)}$ and $\alpha_{(a)}$, for the primary constraints $\tilde{\pi}^{\vec{\varphi}}_{(a)} \approx 0$, ${}^3\tilde{M}_{(a)} \approx 0$ [which do not generate secondary constraints]: they fix the orientation of the tetrads ${}^4E^{(\alpha)}_A$ in every point [the gauge fixings on the $\varphi_{(a)}$'s are equivalent to choose the (in general non-geodesic) congruence of timelike worldlines with 4-velocity field $u^A = {}^4E^A_{(o)}$ corresponding to local observers either at rest or Lorentz boosted; the gauge fixings on the $\alpha_{(a)}$'s are equivalent to the fixation of the standard of non rotation of the local observer];

ii) 3 gauge-fixings for the parameters ξ^r of the spatial pseudodiffeomorphisms generated by the secondary constraints ${}^3\tilde{\Theta}_r \approx 0$: they correspond to the choice of an atlas of coordinates on Σ_τ [chosen as conventional origin of pseudodiffeomorphisms and influencing the parametrization of the angles $\alpha_{(a)}$] and, therefore, by adding the parameter τ, labelling the leaves of the foliation, of an atlas on M^{3+1}. The gauge-fixings on ξ^r, whose time constancy produces the gauge-fixings for the shift functions and, therefore, a choice of simultaneity convention in M^4 (the choice of how to synchronize clocks), can be interpreted as a fixation of 3 standards of length by means of the choice of a coordinate system on Σ_τ;

iii) a gauge-fixing for ρ, which, being a momentum, carries an information about the extrinsic curvature of Σ_τ embedded in M^4 [it replaces the York extrinsic time ${}^3K$ of the Lichnerowicz-York conformal approch] for the superhamiltonian constraint. The gauge-fixing on ρ has nothing to do with a standard of time (the evolution is parametrized by the parameter τ of the induced coordinate system $(\tau, \vec{\sigma})$ on M^4; see also Ref. [6]), but it is a statement about the extrinsic curvature of a Σ_τ embedded in M^4 [the Poisson algebra of the superhamiltonian and supermomentum constraints reflects the embeddability properties of Σ_τ; the superhamiltonian constraint generates the deformations normal to Σ_τ, which partially 'replace' the τ-diffeomorphisms] and is one of the sources of the gauge dependence at the kinematical level of the curvature scalars of M^4 [the other sources are the lapse and shift functions and their gradients]. The natural interpretation of the gauge transformations generated by the superhamiltonian constraint is to change the 3+1 splitting of M^4 by varying the gauge variable $\rho(\tau, \vec{\sigma})$ [i.e. something in the extrinsic curvature of the leaves Σ_τ of the associated foliation], so to make the theory independent from the choice of the original 3+1 splitting of M^4, as it happens with parametrized Minkowski theories. However, since the time constancy of the gauge-fixing on ρ determines the gauge-fixing for the lapse function [which says how the Σ_τ are packed in M^4], there is a connection with the choice of the standard of local proper time. Let us remark that only the gauge-fixing $\rho(\tau,\vec{\sigma}) \approx 0$ [implying ${}^3\vec{K}(\tau,\vec{\sigma}) \approx 0$ only in absence of matter and of gravitational field] leaves the Dirac observables $r_{\bar{a}}$, $\pi_{\bar{a}}$, canonical; with other gauge-fixings the canonical degrees of freedom of the gravitational field have to be redefined.

Therefore, according to constraint theory, given an atlas on a 3+1 splitting M^{3+1} of M^4, the phase space content of the 8 nondynamical Einstein equations is equivalent to the determination of the Dirac observables (namely a kinematical gravitational field not yet solution of the 2 dynamical Einstein equations, i.e. of the final Hamilton equations with the ADM energy as Hamiltonian, see Ref. [6]), whose functional form in terms of the original variables depends on choice of the atlas on M^{3+1} and on a certain part of the extrinsic curvature of Σ_τ.

Let us define a "Hamiltonian kinematical gravitational field" as the quotient of the set of Lorentzian spacetimes $(M^{3+1}, {}^4g)$ with a 3+1 splitting with respect to the Hamiltonian gauge group $\tilde{\mathcal{G}}$ with 14 (8 in metric gravity) generators $[Riem\, M^{3+1}/\tilde{\mathcal{G}}]$: while space diffeomorphisms in $Diff\, M^{3+1}$ coincide with those in $Diff\, \Sigma_\tau$, the "τ-diffeomorphisms" in $Diff\, M^{3+1}$ are replaced by the 5 gauge freedoms associated with ρ, N and $N_{(a)}$.

A representative of a "Hamiltonian kinematical gravitational field" in a given gauge equivalence class is parametrized by $r_{\bar{a}}$, $\pi_{\bar{a}}$ and is an element of a gauge orbit Γ spanned by the gauge variables $\varphi_{(a)}$, $\alpha_{(a)}$, ξ^r, ρ, N, $N_{(a)}$. Let us consider the reduced gauge orbit $\Gamma^{'}$ obtained from Γ by going to the quotient with respect to $\varphi_{(a)}$, $\alpha_{(a)}$, ξ^r. The solution $\phi = e^{q/2}$ of the reduced Lichnerowicz equation is ρ-dependent, so that the gauge orbit $\Gamma^{'}$ contains one conformal 3-geometry (conformal gauge orbit; see the end of Appendix C), or a family of conformal 3-metrics if the ρ-dependence of the solution ϕ does not span all the Weyl rescalings. In addition $\Gamma^{'}$ contains the lapse and shift functions. Now, each 3-metric in the conformal gauge orbit has a different 3-Riemann tensor and different 3-curvature scalars. Since 4-tensors and 4-curvature scalars depend : i) on the lapse and shift functions (and their gradients); ii) on ρ both explicitly and implicitly through the solution of the Lichnerowicz equation, as shown in Appendices A and B of I and in Appendix E in the 3-orthogonal gauges (with the corresponding 3-tensors given in Appendix D), and this influences the 3-curvature scalars, most of these objects are in general gauge variables from the Hamiltonian point of view at least at the kinematical level. The simplest relevant scalars of $Diff\, M^4$, where to visualize these effects, are Komar-Bergmann's individuating fields (see later on) and/or the bilinears ${}^4R_{\mu\nu\rho\sigma}\, {}^4R^{\mu\nu\rho\sigma}$, ${}^4R_{\mu\nu\rho\sigma}\, \epsilon^{\mu\nu\alpha\beta}\, {}^4R_{\alpha\beta}{}^{\rho\sigma}$. Therefore, generically the elements of the gauge orbit $\Gamma^{'}$ are, from the point of view of M^4 based on the Hilbert action, associated with different 4-metrics belonging to different 4-geometries (the standard "kinematical gravitational fields").

According to the gauge interpretation based on constraint theory, a "Hamiltonian kinematical gravitational field" is an equivalence class of 4-metrics modulo the pullback of the Hamiltonian group of gauge transformations, which contains all the 4-geometries connected by them and a well defined conformal 3-geometry. This is a consequence of the different invariance properties of the ADM and Hilbert actions, even if they generate the same equation of motion.

Let us define an ' Hamiltonian Einstein or dynamical gravitational field" as a Hamiltonian kinematical gravitational field which satisfies the final Hamilton equations with the ADM energy as Hamiltonian (equivalent to the two dynamical equations hidden in the Einstein equations).

These Hamiltonian dynamical gravitational fields correspond to special gauge equivalence classes, which contain only one 4-geometry whose representative 4-metrics satisfy Einstein's equations, so that they "coincide" with the standard dynamical gravitational fields. This highly nontrivial statement is contained in the results of Refs. [84,86,85] (in particular see Ref. [86] for the noncompact asymptotically free at spatial infinity case). The implication of this fact is that on the space of the solutions of the Hamilton-Dirac equations (which, together with the first class constraints, are equivalent to Einstein's equations) the kinematical Hamiltonian gauge transformations are restricted to be dynamical symmetries (maps of solutions onto solutions; with them there is not necessarily an associated constant of the motion like with the Noether symmetries of an action) of Einstein's equations in the ADM presentation and this implies that the allowed Hamiltonian gauge transformations must be equivalent to or contained in the spacetime pseudodiffeomorphisms of M^4 (which are dynamical symmetries of Einstein's equations as already said). The allowed infinitesimal Hamiltonian gauge transformations on the space of solutions of the Hamilton-Dirac equations must be solutions of the Jacobi equations (the linearized constraints and the linearized evolution equations; see Refs. [85] for their explicit expression) and this excludes most of the kinematically possible Hamiltonian gauge transformations (all those generating a transition from a 4-geometry to another one). In the allowed Hamiltonian gauge transformations the gauge parameters N, $N_{(a)}$, ρ,... are not independent but restricted by the condition that the resulting gauge transformation must be a spacetime pseudodiffeomorphisms. However, since the infinitesimal spacetime pseudodiffeomorphisms of a 4-metric solution of Einstein's equations (i.e. $\mathcal{L}_{\xi^\rho \partial_\rho}\, {}^4g_{\mu\nu}(x)$) are solutions to the Jacobi equations in the Hilbert form, it turns out that among the dynamical symmetries of Einstein's equations there are both allowed strictly Hamiltonian gauge trasformations, under which the ADM action is quasi-invariant, and generalized transformations under which the ADM action is not invariant (see Appendix A of the next paper [6]). This derives from the fact that the Noether symmetries of an action and the dynamical symmetries of its Euler-Lagrange equations have an overlap but do not coincide. This is the way in which on the space of solutions of Einstein's equations spacetime diffeomorphisms are rebuild starting from the allowed Hamiltonian gauge transformations adapted to the 3+1 splittings of the ADM formalism. The kinematical freedom of the 8 independent types of Hamiltonian gauge transformations of metric gravity is reduced to 4 dynamical types like for $Diff\, M^4$; partially, this was anticipated at the kinematical level by the fact that in the original Dirac Hamiltonian there are only 4 arbitrary Dirac multipliers, and that the gauge-fixing procedure starts with the gauge fixings of the secondary constraints, which generate those for the primary ones , which in turn lead to the determination of the Dirac multipliers.

This state of affairs implies also that the Dirac observables (namely the invariants under the kinematical Hamiltonian gauge transformations and without any a priori tensorial characted under $Diff\, M^4$) restricted to the solutions of the final Hamilton-Dirac equations (and therefore of the original Einstein's equations) must be expressible in some way in terms of quantities scalar under $Diff\, M^4$ when M^4 is an Einstein spacetime. A step in this direction would be to find the connection of our Dirac observables $r_{\bar{a}}(\tau,\vec{\sigma})$ in the 3-orthogonal gauges with the symmetric traceless 2-tensors on 2-planes, which are the independent gravitational degrees of freedom according to Christodoulou and Klainermann [23], and with the in some way connected Newman-Penrose formalism.

In generally covariant theories (without background fields) the interpretational difference with respect to the Dirac observables of Yang-Mills theories, is that one has to make a complete gauge-fixing to give a meaning to "space and time" (in the above sense) before being able to identify the functional form of the Dirac observables for the gravitational field and moreover we have to formulate the problem only for the solutions of Einstein's equations (this is not necessary for Yang-Mills theory).

This deep difference between the interpretations based on constraint theory and on general covariance respectively is reflected in the two viewpoints about what is observable in general relativity (and, as a consequence, in all generally covariant theories) as one can clearly see in Ref. [81] and in its bibliography:

i) The "non-local point of view" of Dirac [29], according to which determinism implies that only gauge-invariant quantities (Dirac's observables; they do not exist globally for compact spacetimes) can be measured. The "hole argument" of Einstein [89] (see Refs. [81,78] for its modern treatment) supports this viewpoint: points of spacetime are not a priori distinguishable (their individuality is washed out by general covariance, i.e. by the invariance under spacetime diffeomorphisms), so that, for instance, ${}^4R(\tau,\vec{\sigma})$ [a scalar under diffeomorphisms, but not a Dirac observable at the kinematical level] is not an observable quantity. Even if ${}^4R(\tau,\vec{\sigma}) \stackrel{\circ}{=} 0$ in absence of matter, the other curvature scalars are non vanishing after having used Einstein equations and, due to the lack of known solutions without Killing vectors, it is not possible to say which is their connection with Dirac observables. More in general, the 4-metric tensor ${}^4g_{\mu\nu}$ is a not observable gauge variable. As said in Ref. [78] an Einstein spacetime manifold with a metric corresponds to a dynamical gravitational field, but a dynamical gravitational field corresponds to an equivalence class of spacetimes. The metrical structure forms part of the set of dynamical variables, which must be determined before the points of spacetime have any physical properties. Therefore, one cannot assume in general relativity what is valid in special relativity, namely that the individuation of the points of Minkowski spacetime is established by a framework of rigid rods and clocks.

In Appendix E this is clearly shown at the kinematical level (i.e. before the restriction to the solutions of Einstein's equations) in the 3-orthogonal gauges: there is the explicit dependence of the 4-tensors on $(M^4, {}^4g)$ on the residual gauge variables (to be fixed to have a reconstruction of M^4 and, therefore, a coordinate system on it) $N = N_{(as)} + n$, $N_{(a)} = {}^3\hat{e}^r_{(a)}[N_{(as)r} + n_r]$, ρ, and on the conformal factor q of the 3-metric, which has to be determined by the superhamiltonian constraint (and this will introduce an extra dependence on the last gauge variable, its conjugate momentum ρ; this is the only gauge freedom of the 3-tensors on $(\Sigma_\tau, {}^3g)$ given in Appendix D). Instead in the Appendices A and B of I there is shown the general gauge dependence of 4-tensors on all the gauge variables before choosing a coordinate system.

Fixing the gauge freedoms in general relativity means to determine the functional form of the 4-metric tensor ${}^4g_{\mu\nu}$: this is a definition of the angle and distance properties of the material bodies, which form the reference system (rods and clocks). At the kinematical level the standard procedures of defining measures of length and time [31,63] are gauge dependent, because the line element ds^2 is gauge dependent and determined only after a complete gauge-fixing and after the restriction to the solutions of Einstein's equations (note that in textbooks these procedures are always defined without any reference to Einstein's equations):

only now the curvature scalars of M^4 become measurable, like the electromagnetic vector potential in the Coulomb gauge. The measuring apparatuses should also be described by the gauge invariant Dirac observables associated with the given gauge (namely identified by the Shanmugadhasan canonical transformation associated with that gauge) as we shall try to show in Refs. [6,30], after the introduction of matter, since an experimental laboratory corresponds by definition to a completely fixed gauge.

See also Ref. [90] for the relevance of the "hole argument" in the discussions on the nature of spacetime and for the attempts to formulate quantum gravity. Even if the standard canonical (either metric or tetrad) gravity approach presents serious problems in quantization due to the intractable Lichnerowicz equation (so that research turned towards either Ashtekar's approach or superstring theory with its bigger general covariance group), still the problem of what is observable at the classical level in generally covariant theories is open.

ii) The 'local point of view', according to which the spacetime manifold M^4 is the manifold of physically determined 'events' (like in special relativity), namely spacetime points are physically distinguishable, because any measurement is performed in the frame of a given reference system. The gauge freedom of generally covariant theories reflects the freedom of choosing coordinate systems, i.e. reference systems. Therefore, the evolution is not uniquely determined (since the reference systems are freely chosen) and, for instance, ${}^4R(\tau, \vec{\sigma})$ is an observable quantity, like the 4-metric tensor ${}^4g_{\mu\nu}$. See Ref. [91] for a refusal of Dirac's observables in general relativity based on the local point of view.

In Ref. [81] the non-local point of view is accepted and there is a proposal for using some special kind of matter to define a "material reference system" (not to be confused with a coordinate system) to localize points in M^4, so to recover the local point of view in some approximate way [the main approximations are: 1) to neglect, in Einstein equations, the energy-momentum tensor of the matter forming the material reference system (it's similar to what happens for test particles); 2) to neglect, in the system of dynamical equations, the entire set of equations determining the motion of the matter of the reference system (this introduces some indeterminism in the evolution of the entire system)], since in the analysis of classical experiments both approaches tend to lead to the same conclusions. See also Refs. [82,60,61] for a complete review of material clocks and reference fluids. However, we think that one has to consider the use of "test objects" as an idealization for the attempt to approximate with realistic dynamical objects the conformal, projective, affine and metric structures [80] of Lorentzian manifolds, which are used to define the "ideal geodesic clocks" [63] and the basis of the theory of measurement.

Let us remark that in applications, for instance in the search of gravitational waves, one is always selecting a background reference metric and the associated (Minkowski like) theory of measurement: the conceptual framework becomes the same as in special relativity. The same happens for every string theory due to necessity (till now) of a background metric in their formulation.

Since in Refs. [6,30] we shall present a different solution for the time problem (in a scheme in which a "mathematical time" is identified before quantization and never quantized), we delay the discussion of these problems to these papers. In them the deterministic evolution of general relativity, in the mathematical parameter τ labelling the leaves Σ_τ of the foliation (to be locally correlated to some physical time), is generated by the ADM energy (see also Ref. [86]) and there is a decoupled "point particle clock" measuring τ. Let us remark that the refuse of internal either intrinsic or extrinsic times implies that the superhamiltonian constraint has to be interpreted as a generator of gauge transformations [so that the momentum ρ, conjugate to the conformal factor q of the 3-metric, is a gauge variable] and not as a generator of time evolution, contrary to the commonly accepted wiewpoint for compact spacetimes (see Kuchar in Ref. [92]) .

Instead, we accept the proposal of Komar and Bergmann [93,94] of identifying the physical points of a spacetime $(M^4, {}^4g)$ without Killing vectors, solution of the Einstein's equations, only a posteriori in a way invariant under spacetime diffeomorphisms extended to 4-tensors, by using four invariants bilinear and trilinear in the Weyl tensors [as shown in Ref. [95] there are 14 algebraically independent curvature scalars for M^4, which are reduced to four when Einstein equations without matter are used], called "individuating fields", which do not depend on the lapse and shift functions. These individuating fields depend on $r_{\bar{a}}$, $\pi_{\bar{a}}$ and on the gauge parameters ξ^r (choice of 3-coordinates on Σ_τ) and ρ (replacing York's internal extrinsic time ${}^3K$): note the difference from the proposal of Refs. [60,62] of using ξ^r and q for this aim. The 4-metric in this "physical 4-coordinate grid", obtained from ${}^4g_{AB}$ by making a coordinate transformation from the adapted coordinates $\sigma^A = (\tau, \vec{\sigma})$, depends on the same variables and also on the lapse and shift functions.

By using Appendices A, B of I and E, one can see that these individuating fields are not Dirac observables at the kinematical level. They must not be Dirac observables also when restricted to the solutions of Einstein's equations, because the freedom in the choice of the mathematical coordinates σ^A is replaced by the gauge freedom in the choice of ξ^r and ρ. However, in every complete gauge (choice of the coordinate systems on Σ_τ and on M^{3+1}) they describe a special gauge-dependent coordinate system for M^4, in which the dynamical gravitational field degrees of freedom in that gauge can be used (at least in some finite region) to characterize distinct points of M^4, as also remarked by Stachel [78] in connection with Einstein's hole argument [but without taking into account constraint theory]. In this way we get a physical 4-coordinate grid on the mathematical 4-manifold M^4 dynamically determined by tensors over M^4 itself with a rule which is invariant under $Diff\, M^4$ but with the functional form of the map "$\sigma^A = (\tau, \vec{\sigma}) \mapsto physical\, 4 - coordinates$ depending on the chosen complete gauge: the "local point of view" is justified a posteriori in every completely fixed gauge.

Finally, let us remember that Bergmann [94] made the following critique of general covariance: it would be desirable to restrict the group of coordinate transformations (spacetime diffeomorphisms) in such a way that it could contain an invariant subgroup describing the coordinate transformations that change the frame of reference of an outside observer (these transformations could be called Lorentz transformations; see also the comments in Ref. [31] on the asymptotic behaviour of coordinate transformations); the remaining coordinate transformations would be like the gauge transformations of electromagnetism. This is what we began to do in Section II with the redefinition of lapse and shift functions and which will be completely accomplished in the next papers [6,30] on Poincaré charges and on the deparametrization of tetrad gravity in presence of matter. In this way "preferred" asymptotic coordinate systems will emerge [see the preferred congruence of asymptotic timelike observers in Ref. [6]], which, as said by Bergmann, are not "flat": while the inertial coordinates are determined experimentally by the observation of trajectories of force-free bodies, these intrinsic coordinates can be determined only by much more elaborate experiments (for instance precessional effects on gyroscopes), since they depend, at least, on the inhomogeneities of the ambient gravitational fields.

See also Ref. [96] for other critics to general covariance: very often to get physical results one uses preferred coordinates not merely for calculational convenience, but also for understanding. In Ref. [97] this fact has been formalized as the "principle of restricted covariance". In our case the choice of the gauge-fixings has been dictated by the Shanmugadhasan canonical transformations, which produce generalized Coulomb gauges, in which one can put in normal form the Hamilton equations for the canonical variables of the gravitational field [and, therefore, they also produce a normal form of the two associated combinations of the Einstein equations which depend on the accelerations].

This discussion points towards the necessity of finding suitable weighted Sobolev spaces such that: i) there are no isometries of the metric (Gribov ambiguities of the spin connection); ii) there are no supertranslations; iii) Poincaré charges at spatial infinity are well defined; iv) there is a well defined Hamiltonian group $\bar{\mathcal{G}}$ of gauge transformations which preserves properties i), ii) and iii). It is hoped that its pullback $\tilde{\mathcal{G}}$, acting on tensors on M^{3+1}, will contain asymptotic Poincaré transformations as an invariant subgroup (implying the existence of Bergmann's "preferred" coordinate systems). These problems will be studied in the next paper [6].

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APPENDIX A: SPECIAL SYSTEMS OF COORDINATES.

The gauge freedom of general relativity, due to its invariance under general coordinate transformations or spacetime diffeomorphisms, reflects the arbitrariness in the choice of how to describe space and time since coordinates have no intrinsic meaning. The choice of a local coordinate system is equivalent in standard presentations to the definition of an observer with his ideal clocks and rods and the principle of general covariance states that the laws of physics are independent from this choice. However in trying to solve Einstein partial differential equations or to find local canonical adapted bases with the Shanmugadhasan canonical transformation one has to look for those coordinate systems (if any) which separate the equations. Therefore, the choice of adapted bases and probably also some future definition of elementary particle in general relativity (so that the standard Wigner definition will re-emerge in the limit of flat Minkowski spacetime) require a breaking of general covariance. At least locally one has to choose "physical coordinate systems adapted to the physical systems under investigation", study there the equations of motion and then use general covariance in a passive way as mathematical coordinate transformations, which possibly can transform localized concepts in spacetime delocalized ones.

While the weak (or Galilei) form of the equivalence principle (implying the equality of inertial and gravitational masses) is common to Newton and Einstein gravity [the laws of motion of free particles in a local, freely falling, nonrotating frame are identical to Newton's laws of motion expressed in a gravity-free Galilean frame: they will produce straight worldlines in a local Lorentz frame (i.e. in a freely falling nonrotating frame) as in special relativity, in absence of electric charge, for small angular momentum, for gravitational binding energies much less of rest masses and in a sufficiently small neighbourhood such that the effects of the geodesic deviation equation are negligible], the Einstein medium-strong and strong forms assert the existence of local Lorentz frames for all the nongravitational laws and all the laws of physics respectively. In particular, the strong form implies that there are no gravitational effects in a local freely falling nonrotating frame in a sufficiently small spacetime neighbourhood [63,75] in which tidal effects are negligible.

"Ideal" rods and clocks are defined as being ones which measure proper length $\triangle s = \sqrt{-\epsilon^4 g_{\mu\nu}\triangle^\mu\triangle^\nu}$ ($\triangle^\mu$ spacelike) or proper time $\triangle\tau = \sqrt{\epsilon^4 g_{\mu\nu}\triangle^\mu\triangle^\nu}$ ($\triangle^\mu$ timelike); one must then determine the accuracy to which a given rod or clock is ideal under given circumstances by using laws of physics to analyze its behaviour [63] (see the Conclusions for the interpretational problems).

See Refs. [98,63] for a review of relevant coordinate systems [Ref. [98] uses $\epsilon = +1$].

We shall add only some informations about geodesic coordinates, harmonic coordinates and holonomic versus nonholonomic coordinates. See the previous references and Appendix A of I for coordinates 1) geodesic along a specified curve [which include Fermi-Walker and Fermi transport of tetrads (gyroscopes) and Fermi normal coordinates]; 2) semigeodesic [which include Gaussian normal (or synchronous) coordinates]].

A)" Coordinates x_1^μ geodesic (or locally inertial) at a point $p \in M^4$" chosen as origin $x_1^\mu|_p = 0$. They are such that ${}^4\Gamma^\alpha_{\beta\gamma}(x_1 = 0) = 0$, so that at p the geodesic equation is $\frac{d^2}{d\tau^2} x_1^\mu = 0$.

Aa)" Local Lorentzian (or Minkowskian or inertial or comoving) frames". An observer falling freely in M^4 makes measurements in his local Lorentz frame, defined by

$$
{}^4g_{\mu\nu}(x_1=0) = {}^4\eta_{(\mu)(\nu)}, \qquad \partial_\alpha {}^4g_{\mu\nu}|_{x_1=0} = 0, \Rightarrow {}^4\Gamma^\alpha_{\beta\gamma}(x_1=0) = 0. \tag{A1}
$$

The observer is at rest in his local Lorentz frame, i.e. his worldline is $\{x_1^o\, arbitrary,\, x_1^k = 0\}$: his velocity is $u_1^\mu = \frac{dx_1^\mu}{d\tau}|_{x_1^k=0} = {}^4\nabla_{u_1}\, u_1^\mu|_{x_1^k=0} = ({}^4\nabla_{u_1^o} u_1^o|_{x_1^k=0}, \vec{0}) = u_1^o\, {}^4\Gamma^\mu_{oo}\, u_1^o|_{x_1^k=0} = 0$. Therefore the observer is freely falling since he moves along a geodesic [${}^4\nabla_{u_1} u_1^\mu = 0$; in the local Lorentz frame the geodesic is an extremal of proper time, $d\tau = \sqrt{\epsilon^4\eta_{(\alpha)(\beta)} dx_1^{(\alpha)} dx_1^{(\beta)}}$ and has no acceleration, $a_1^\mu = \frac{d}{d\tau} u_1^\mu = 0$]. Local Lorentz frames remains geodesic at $p \in M^4$ under linear transformations of coordinates, because the Christoffel symbols behave like tensors under such transformations.

Ab) "Riemann coordinates x_2^μ geodesic at $p \in M^4$", where $x_2^\mu|_p = 0$. Their distinctive feature is that geodesics passing through $p \in M^4$ satisfy the same equations for straight lines passing through p as in Euclidean geometry with Cartesian coordinates: if τ is an affine parameter along anyone of these geodesics ($\tau = 0$ in p), the geodesics through p have the form $x_2^\mu(\tau) = \xi^\mu \tau$, where $\xi^\mu = \frac{dx_2^\mu(\tau)}{d\tau}|_p$ is the tangent to the geodesic at p (it is constant along the geodesic). Riemann coordinate systems at $p \in M^4$ are related by linear homogeneous transformations, which preserve the form of the geodesics. They exists in a neighbourhood V of $p \in M^4$, in which the geodesics emanating from it do not cross each other [99] inside V, so that p can be joined to each point of V by means of them (V is geodesically complete). The singularity theorems tend to say that every pseudo-Riemannian M^4 cannot be geodesically complete; instead Riemannian 3-manifolds like Σ_τ (only spacelike geodesics) may be geodesically complete.

Necessary and sufficient criteria defining Riemannian coordinates at $p \in M^4$ are:

(α) The geodesics through $p \in M^4$ have the form $x_2^\mu(\tau) = \xi^\mu \tau$, with $\xi^\mu = const.$.

(β) The Christoffel symbols satisfy ${}^4\Gamma^\alpha_{\mu\nu}(x_2) = 0$ (i.e. Riemann coordinates are geodesic at p), so that the equation for the geodesics through p is $\frac{d^2 x_2^\mu(\tau)}{d\tau^2} = 0$.

(γ) If ${}^4g_{\mu\nu}$ are analytic functions in a neighbourhood of p, the following symmetrized derivative of the Christoffel symbols vanish at p: $\partial_{(\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_r}\, {}^4\Gamma^\alpha_{\nu_1\nu_2)}(x_2=0) = 0$, and one has [similar expansions hold for every tensor]:

$$
\begin{aligned}
{}^4g_{\mu\nu}(x_2) &= {}^4g_{\mu\nu}(0) - \frac{1}{3}{}^4R_{\mu\alpha\nu\beta}(0) x_2^\alpha x_2^\beta - \frac{1}{3!}\partial_\rho\, {}^4R_{\mu\alpha\nu\beta}|_{x_2=0} x_2^\rho x_2^\alpha x_2^\beta + \\
&+ \frac{1}{5!}[-6\partial_\rho\partial_\sigma\, {}^4R_{\mu\alpha\nu\beta}|_{x_2=0} + \frac{16}{3}{}^4R_{\rho\nu\sigma}{}^\epsilon(0)\, {}^4R_{\beta\mu\alpha\epsilon}(0)] x_2^\rho x_2^\sigma x_2^\alpha x_2^\beta + \cdots .
\end{aligned} \tag{A2}
$$

Ab1) "Riemann normal (or simply normal) coordinates y^μ". It is a special system of Riemann coordinates for which one has ${}^4g_{\mu\nu}(0) = {}^4\eta_{(\mu)(\nu)}$; in it one has

$$
\begin{aligned}
{}^4g &= |det\,({}^4g_{\mu\nu})| = \{1 - \frac{1}{3}{}^4R_{\mu\nu}(0) x_2^\mu x_2^\nu - \frac{1}{3!}\partial_\rho\, {}^4R_{\mu\nu}|_{x_2=0} x_2^\rho x_2^\mu x_2^\nu - \\
&- \frac{1}{4!}[\frac{4}{3}{}^4R_{\mu\nu}(0)\, {}^4R_{\alpha\beta}(0) + \frac{4}{15}{}^4R_{\mu\rho\nu}{}^\lambda(0)\, {}^4R_{\alpha\lambda\beta}{}^\rho(0) + \\
&+ \frac{6}{5}\partial_\mu\partial_\nu\, {}^4R_{\alpha\beta}|_{x_2=0}] x_2^\mu x_2^\nu x_2^\alpha x_2^\beta + ..\},
\end{aligned} \tag{A3}
$$

so that, if ${}^4R_{\mu\nu}(0) = 0$, then ${}^4g = \epsilon[1 + \frac{1}{90}{}^4R_{\mu\rho\nu}{}^\lambda(0)\, {}^4R_{\alpha\lambda\beta}{}^\rho(0) x_2^\mu x_2^\nu x_2^\alpha x_2^\beta + \cdots]$. A normal coordinate system at $p \in M^4$ is defined to within a linear Lorentz transformation.

Normal coordinates exploit to the full the locally Minkowskian properties of pseudo-Riemannian 4-manifolds: an observer who assigns coordinates in the neighbourhood of a given event $p \in M^4$ by theodolite measurements at p and interval measurements from p as if spacetime were flat, will assign normal coordinates; i.e. the observer at p [where an inertial observer in using a frame (tetrad) ${}^4E_{1(\alpha)} = {}^4E^{\mu}_{1(\alpha)} \partial/\partial x^{\mu}_1$] fills spacetime near p with geodesics radiating out from p, with each geodesic (with a suitable choice of affine parameter) determined by its tangent vector at p.

Cartan [100,101] showed that, given Riemann normal coordinates y^{μ} at $p \in M^4$ $[y^{\mu}|_p = 0]$, one can choose adapted orthonormal frames and coframes ${}^4E^{(N)}_{(\alpha)} = {}^4E^{(N)\mu}_{(\alpha)}(y) \partial/\partial y^{\mu}$, ${}^4\theta^{(N)(\alpha)} = {}^4E^{(N)(\alpha)}_{\mu}(y) dy^{\mu}$, obtained from ${}^4E^{(N)}_{(\alpha)}|_p = \delta^{\mu}_{(\alpha)} \partial/\partial y^{\mu}$, ${}^4\theta^{(N)(\alpha)}|_p = \delta^{(\alpha)}_{\mu} dy^{\mu}$, by parallel transport along the geodesic arcs originating at p. Then one has the following properties

$$
\begin{aligned}
{}^4E^{(N)(\alpha)}_{\mu}(y)\, y^{\mu} &= \delta^{(\alpha)}_{\mu} y^{\mu} \\
{}^4\theta^{(N)(\alpha)} &= \delta^{(\alpha)}_{\mu} [dy^{\mu} + y^{\rho} y^{\sigma}\, N^{\mu}{}_{\rho\sigma\lambda}(y)\, dy^{\lambda}], \\
N_{\mu\rho\sigma\lambda} &= -N_{\rho\mu\sigma\lambda} = -N_{\mu\rho\lambda\sigma}.
\end{aligned} \tag{A4}
$$

Since normal coordinates are the most natural from a differential geometric point of view, let us look for a parametrization, in this system of coordinates, of the Dirac observables ${}^3\hat{e}_{(a)r}(\tau, \vec{\sigma})$ on Σ_{τ} in terms of 3 real functions $\hat{Q}_r(\tau, \vec{\sigma})$, whose conjugate momenta will be denoted $\hat{\tilde{\Pi}}^r(\tau, \vec{\sigma})$. Eqs.(A4) give the Cartan definition of orthonormal tetrads adapted to normal coordinates for Lorentzian 4-manifolds. This suggest that for Riemannian 3-manifolds like Σ_{τ}, the reduced cotriads ${}^3\hat{e}_{(a)r}(\tau, \vec{\sigma})$ may be parametrized as follows

$$
\begin{aligned}
{}^3\hat{e}_{(a)r}(\tau, \vec{\sigma}) &= \delta^s_{(a)} [\delta_{rs} + \sum_n \epsilon_{run} \epsilon_{svn} \sigma^u \sigma^v \hat{Q}_n(\tau, \vec{\sigma})] \\
&\Rightarrow {}^3\hat{e}_{(a)r}(\tau, \vec{\sigma})\, \sigma^r = \delta_{(a)r}\, \sigma^r,
\end{aligned} \tag{A5}
$$

with $N_{surv}(\tau, \vec{\sigma}) = \sum_n \epsilon_{sun} \epsilon_{rvn} \hat{Q}_n(\tau, \vec{\sigma}) = -N_{usrv}(\tau, \vec{\sigma}) = -N_{suvr}(\tau, \vec{\sigma}) = N_{rvsu}(\tau, \vec{\sigma})$. Then one gets

$$
\begin{aligned}
{}^3\hat{g}_{rs}(\tau, \vec{\sigma}) &= {}^3\hat{e}_{(a)r}(\tau, \vec{\sigma})\, {}^3\hat{e}_{(a)s}(\tau, \vec{\sigma}) = \delta_{rs} + \\
&+ \sigma^u \sigma^v [\sum_n \epsilon_{run} \epsilon_{svn} (2 + \vec{\sigma}^2\, \hat{Q}_n(\tau, \vec{\sigma})) \hat{Q}_n(\tau, \vec{\sigma}) - \sum_{nm} \epsilon_{run} \epsilon_{svm} \sigma^n \sigma^m \hat{Q}_n(\tau, \vec{\sigma}) \hat{Q}_m(\tau, \vec{\sigma})],
\end{aligned} \tag{A6}
$$

to be compared with Eq.(A2).

A special case of normal coordinates, $y^{\mu} = \xi^{\mu}$, (with a special orientation of the tetrads by means of SO(3,1) rotations) is realized, even when torsion is present, with the “radial gauge” [102] (see Ref. [103,104] for a review). To get it, one imposes the gauge conditions

$$
\begin{aligned}
&\xi^{\mu}\, {}^4\omega^{(\alpha)}_{\mu(\beta)}(\xi) = 0 \quad \Rightarrow {}^4\omega^{(\alpha)}_{\mu(\beta)}(0) = 0, \\
&\xi^{\mu}\, {}^4E^{(\alpha)}_{\mu}(\xi) = \delta^{(\alpha)}_{\mu} \xi^{\mu} \quad \Rightarrow {}^4E^{(\alpha)}_{\mu}(0) = \delta^{(\alpha)}_{\mu}, \\
\\
&{}^4\Gamma^{\rho}_{\mu\nu} = {}^4E^{\rho}_{(\alpha)}\, {}^4E^{(\beta)}_{\mu}\, {}^4\omega^{(\alpha)}_{\nu(\beta)} + {}^4E^{\rho}_{(\alpha)} \partial_{\nu}\, {}^4E^{(\alpha)}_{\mu} \Rightarrow \\
&\Rightarrow \xi^{\nu}\, {}^4\Gamma^{\rho}_{\mu\nu} = {}^4E^{\rho}_{(\alpha)}\, \xi^{\nu} \partial_{\nu}\, {}^4E^{(\alpha)}_{\mu} \quad \Rightarrow \xi^{\mu} \xi^{\nu}\, {}^4\Gamma^{\rho}_{\mu\nu} = 0.
\end{aligned} \tag{A7}
$$

The first condition means that the tetrads are parallel transported from the origin $\xi^\mu = 0$ along the straight lines $\xi^\mu(s) = sv^\mu$, $0 \leq s$ and $v^\mu = const.$, while the second one means that these lines are autoparallel, so that in absence of torsion they are geodesics. These equations determine univocally both the coordinate system ξ^μ and the tetrad field ${}^4E^\mu_{(\alpha)}$ in a neighbourhood of the origin [the point with coordinates $\xi^\mu = 0$], so that the gauge conditions are locally attainable and complete. There is a residual global gauge freedom due to the arbitrariness of the choice of the origin and to the possibility of making a Lorentz transformation of the tetrad in the origin.

In Σ_τ-adapted coordinates $\sigma^A = (\tau, \vec{\sigma})$ and with the tetrads ${}^4E^{(\alpha)}_A$ of Eqs. (45), (46) of I, the second condition becomes $\sigma^A\, {}^4E^{(\alpha)}_A(\sigma^d) = \delta^{(\alpha)}_A \sigma^A$, ${}^4E^{(\alpha)}_A(0) = \delta^{(\alpha)}_A$: together with the first condition on the spin 4-connection they determine the gauge variables of tetrad gravity without using explicitly the 3+1 decomposition of the 4-metric.

The radial gauge conditions can be regarded, in a sense, as an operational prescription which permits one to locate the measuring instruments in a neighbourhood of the observer, who lies at the origin $\xi^\mu = 0$. In fact, a simple way to explore this neighbourhood is to send from the origin many "space-probes" carrying clocks, gyroscopes and measuring instruments. A space-probe will be launched with 4-velocity $v^{(\alpha)}$ with respect to the given tetrad ${}^4E^\mu_{(\alpha)}$ and, if τ is the proper time measured by the clock, $\xi^\mu = \tau v^\mu$ are the normal coordinates (in the absence of torsion). Of course, in Minkowski spacetime only the interior of the future cone can be explored in this way.

B) "Harmonic coordinates x^μ_3". Given an arbitrary system of coordinates y^μ, let us consider the wave equation in M^4, ${}^4\Box\varphi(y) = {}^4\nabla_\mu\, {}^4\nabla^\mu\, \varphi(y) = \frac{1}{\sqrt{{}^4g}} \partial_\alpha (\sqrt{{}^4g}\, {}^4g^{\alpha\beta}\, \partial_\beta)\varphi(y) = 0$. The harmonic coordinates are defined as $x^\mu_3 = \varphi^\mu(y)$ with $\varphi^\mu(y)$ four independent solutions of the wave equation, so that the harmonic coordinate condition can be written as $\frac{1}{\sqrt{{}^4g}} \partial_\alpha (\sqrt{{}^4g}\, {}^4g^{\alpha\beta}) = 0$. See Ref. [99] for a discussion of these coordinates, which have to be used for the study of the Cauchy problem for Einstein equations [maximal Cauchy developments, Cauchy stability,...]. If harmonic coordinates hold on an initial data slice, they give a "reduced" form of the Einstein equations that is hyperbolic and preserves both the constraints and the harmonic condition in the evolution.

C) "Nonholonomic coordinates". Given anyone of the previous coordinate systems, one can define "coordinate hypersurfaces" $x^\mu = const.$ and "coordinate lines" on which only one of the x^μ is not fixed. Moreover, since one has local coordinate bases $\partial/\partial x^\mu$ and dx^μ for TM^4 and T^*M^4 respectively, it turns out that the tangent vectors to the x^μ coordinate line are $l_{(\mu)} = \delta^\nu_{(\mu)} \partial/\partial x^\nu$ [we put the index μ inside round brackets to emphasize that it numbers the tangent vectors] with controvariant components $l^\nu_{(\mu)} = \delta^\nu_{(\mu)} = \partial x^\nu/\partial x^\mu$ and that their duals are $\theta^{(\mu)} = \delta^{(\mu)}_\nu dx^\nu$ with covariant components $\theta^{(\mu)}_\nu = \delta^{(\mu)}_\nu = \partial x^\mu/\partial x^\nu$. Therefore, $\theta^{(\mu)}$ are a system of 4 gradient vectors : $\theta^{(\mu)}_\nu = \partial_\nu x^\mu$, $\partial_\rho \theta^{(\mu)}_\nu - \partial_\nu \theta^{(\mu)}_\rho = 0$.

A system of coordinates is said "holonomic" if the basic covariant vectors are a field of gradient vectors. When this does not happen, the coordinate system is said "nonholonomic" (it can only be defined in a region of M^4 which can be shrunk to a point): in this case one has local dual noncoordinate bases [tetrads and cotetrads, frames and coframes,..; see Section II of I] ${}^4\hat{E}_{(\mu)} = {}^4\hat{E}^\nu_{(\mu)}(x)\partial/\partial x^\nu$ and ${}^4\hat{\theta}^{(\mu)} = {}^4\hat{E}^{(\mu)}_\nu(x) dx^\nu$ with ${}^4\hat{E}^\nu_{(\alpha)}\, {}^4\hat{E}^{(\beta)}_\nu = \delta^{(\beta)}_{(\alpha)}$. Nonholonomic noncoordinate bases may be chosen orthogonal ${}^4\hat{E}^{(\alpha)}_\mu\, {}^4\hat{E}^\nu_{(\alpha)} = \delta^\nu_\mu$. The basic covariant vectors ${}^4\hat{\theta}^{(\mu)}$ of a nonholonomic coordinate system are a nonintegrable field of tangents to 4 coordinate lines, which define coordinates $x^{(\mu)}$ [$dx^{(\mu)} \neq {}^4\hat{\theta}^{(\mu)}$ since ${}^4\hat{E}^{(\mu)}_\nu$ are not gradients]. In Ref. [105] it is shown how to construct a local nonholonomic coordinate system $x^{(\mu)}$ from local holonomic ones y^μ defined in a neighbourhood of a point $p \in M^4$ [chosen as origin: $y^\mu|_p = 0$], with associated orthonormal frames and coframes ${}^4E_{(\alpha)} = {}^4E^\mu_{(\alpha)}(y)\partial/\partial y^\mu$, ${}^4\theta^{(\alpha)} = {}^4E^{(\alpha)}_\mu(y)dy^\mu$. In the neighbourhood of p, where the coframe matrix ${}^4E^{(\alpha)}_\mu$ is regular, the new coordinates are $x^{(\alpha)} = {}^4E^{(\alpha)}_\mu(y)y^\mu$, so that $y^\mu = {}^4E^\mu_{(\alpha)}(x(y))x^{(\alpha)}$.

APPENDIX B: ISOMETRIES AND CONFORMAL TRANSFORMATIONS.

See Section II of I for the notations and Refs. [106,99].

A diffeomorphism $\phi \in Diff\, M^4$ is an "isometry" of a pseudo-Riemannian manifold $(M^4, {}^4g)$ if ${}^4g^{'}_{\mu\nu}(x^{'}(x)) = {}^4g_{\mu\nu}(x)$ with $x^{'\mu} = \phi^\mu(x)$ [it is called isometry since it preserves the length of a vector; ϕ can be interpreted (in the active sense) as a rigid motion]. The isometries of a given 4-manifold form a group. The infinitesimal isometries $x^{'\mu}(x) = x^\mu + \xi^\mu(x) = x^\mu + \delta_o x^\mu = x^\mu + \epsilon X_\phi x^\mu$ [ϵ is an infinitesimal parameter] are generated by a vector field X_ϕ called a "Killing vector field", satisfying the equation

$$[\mathcal{L}_{X_\phi}{}^4g]_{\mu\nu} = X^\rho_\phi \partial_\rho\, {}^4g_{\mu\nu} + \partial_\mu X^\rho_\phi\, {}^4g_{\rho\nu} + \partial_\nu X^\rho_\phi\, {}^4g_{\mu\rho} = 0,$$

which becomes

$$({}^4\nabla_\mu X_\phi)_\nu + ({}^4\nabla_\nu X_\phi)_\mu = \partial_\mu X_{\phi\,\nu} + \partial_\nu X_{\phi\,\mu} - 2\,{}^4\Gamma^\lambda_{\mu\nu} X_{\phi\,\lambda} = 0$$

[Killing equation] with the Levi-Civita connection. The Killing vector fields of a 4-manifold span the Lie algebra of the isometry group.

A diffeomorphism of $(M^4, {}^4g)$ such that ${}^4g^{'}_{\mu\nu}(x^{'}(x)) = e^{2\sigma(x)}\,{}^4g_{\mu\nu}(x)$ is a "conformal isometry" of the 4-manifold, namely a particular conformal transformation [it changes the scale but not the shape; the angles are preserved]. In general, conformal transformations [$x^{'\mu}(x)$ such that ${}^4g^{'}_{\mu\nu}(x^{'}(x)) = e^{2\sigma(x)}\,{}^4g_{\mu\nu}(x)$] are not spacetime diffeomorphisms; the conformal transformations form a group, $Conf\, M^4$, and the conformal isometries are conformal transformations in $Diff\, M^4 \cap Conf\, M^4$, the only conformal transformations under which Einstein metric gravity is invariant. The conformal transformation preserve the time-, light- (or null-) and space-like character of the objects. One has the following effects of conformal transformations

$$\begin{aligned}
{}^4g_{\mu\nu}(x) &\mapsto {}^4g^{'}_{\mu\nu}(x^{'}(x)) = e^{2\sigma(x)}\,{}^4g_{\mu\nu}(x),\\
{}^4\Gamma^\mu_{\alpha\beta}(x) &\mapsto {}^4\Gamma^{'\mu}_{\alpha\beta}(x^{'}(x)) = {}^4\Gamma^\mu_{\alpha\beta}(x) + \delta^\mu_\beta \partial_\alpha\sigma(x) + \delta^\mu_\alpha \partial_\beta \sigma(x) - {}^4g_{\alpha\beta}(x)\,{}^4g^{\mu\nu}(x)\partial_\nu\sigma(x),\\
{}^4R^\mu{}_{\alpha\nu\beta}(x) &\mapsto {}^4R^{'\mu}{}_{\alpha\nu\beta}(x^{'}(x)) = {}^4R^\mu{}_{\alpha\nu\beta}(x) - {}^4g_{\alpha\beta}(x)\,{}^4B^\mu_\nu(x) - {}^4g_{\alpha\nu}(x)\,{}^4B^\mu_\beta(x) +\\
&\quad + {}^4g_{\alpha\rho}(x)[{}^4B^\rho_\nu(x)\delta^\mu_\beta - {}^4B^\rho_\beta(x)\delta^\mu_\nu],\\
{}^4R_{\mu\nu}(x) &\mapsto {}^4R^{'}_{\mu\nu}(x^{'}(x)) = {}^4R_{\mu\nu}(x) - {}^4g_{\mu\nu}(x)\,{}^4B^\rho_\rho(x) - 2\,{}^4B_{\mu\nu}(x),\\
{}^4R(x) &\mapsto {}^4R^{'}(x^{'}(x)) = e^{-2\sigma(x)}[{}^4R(x) - 6\,{}^4B^\rho_\rho(x)],\\
{}^4C_{\mu\nu\alpha\beta}(x) &\mapsto {}^4C^{'}_{\mu\nu\alpha\beta}(x^{'}(x)) = {}^4C_{\mu\nu\alpha\beta}(x),
\end{aligned} \tag{B1}$$

where ${}^4B^\mu_\nu = -\partial_\nu\sigma\,{}^4g^{\mu\rho}\,\partial_\rho\sigma + {}^4g^{\mu\rho}(\partial_\nu\partial_\rho\sigma - {}^4\Gamma^\gamma_{\nu\rho}\partial_\gamma\sigma) + \frac{1}{2}\delta^\mu_\nu\,{}^4g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma$, ${}^4B_{\mu\nu} = {}^4B_{\nu\mu}$.

Instead for 3-manifolds one has [${}^3\nabla_r$ is the covariant derivative associated with the 3-metric ${}^3g_{rs}$]:

$$\begin{aligned}
&{}^3g_{rs} \mapsto {}^3\hat{g}_{rs} = \phi^4\,{}^3g_{rs},\\
&{}^3\Gamma^u_{rs} \mapsto {}^3\hat{\Gamma}^u_{rs} = {}^3\Gamma^u_{rs} + 2\phi^{-1}(\delta^u_r\,{}^3\nabla_s\phi + \delta^u_s\,{}^3\nabla_r\phi - {}^3g_{rs}\,{}^3g^{uv}\,{}^3\nabla_v\phi),\\
&{}^3R \mapsto {}^3\hat{R} = \phi^{-4}\,{}^3R - 8\phi^{-5}({}^3g^{uv}\,{}^3\nabla_u\,{}^3\nabla_v)\phi.
\end{aligned}$$

Given two 4-metrics ${}^4g$ and ${}^4\bar{g}$ on the same 4-manifold M^4, they are called "conformally related" if ${}^4\bar{g}_{\mu\nu}(x) = e^{2\sigma(x)}\, {}^4g_{\mu\nu}(x)$; the equivalence class of the conformally related 4-metrics on a 4-manifold M^4 is called a "conformal structure". The transformation ${}^4g \mapsto e^{2\sigma}\, {}^4g$ is called a "Weyl rescaling" and the set of Weyl rescalings on M^4 is a group $Weyl\, M^4$.

If every point $x^\mu \in M^4$ of the pseudo-Riemannian manifold $(M^4, {}^4g)$ lies in a coordinate chart where ${}^4g_{\mu\nu} = e^{2\sigma}\, {}^4\eta_{\mu\nu}$, then $(M^4, {}^4g)$ is said to be "conformally flat"; the vanishing of the Weyl tensor ${}^4C_{\mu\nu\alpha\beta}$ is the necessary and sufficient condition for conformal flatness.

An infinitesimal conformal isometry is generated by a so called "conformal Killing vector field" X, which satisfies the conformal Killing equation

$$X^\rho \partial_\rho {}^4g_{\mu\nu} + \partial_\mu X^\rho {}^4g_{\rho\nu} + \partial_\nu X^\rho {}^4g_{\mu\rho} = \frac{1}{4} {}^4g_{\mu\nu} [X^\rho {}^4g^{\alpha\beta} \partial_\rho {}^4g_{\alpha\beta} + 2\partial_\rho X^\rho].$$

The dilatation vector field $D = x^\mu \partial_\mu$ is a conformal vector field of Minkowski spacetime. Since $X_\mu n^\mu$, with n^μ tangent to a geodesic γ, is constant only for null geodesics $[n^\mu n_\mu = 0]$, conformal Killing vector fields give rise to constants of motion for light rays.

APPENDIX C: THE LICHNEROWICZ-YORK CONFORMAL APPROACH TO THE CANONICAL REDUCTION OF METRIC GRAVITY.

To give an idea of the Lichnerowicz-York conformal approach to canonical reduction [74,65] [see Refs. [66,55,75] for reviews], we need some preliminary concepts. See Section V of I for a review of ADM canonical metric gravity.

A) Since the hypersurfaces $\mathcal{T} = -\frac{4}{3}\epsilon k\, {}^3K = const.$ of constant mean extrinsic curvature [CMC slices] play an important role for the reduction of Hamiltonian constraints in the conformal approach [${}^3K(\tau, \vec{\sigma}) - const. \approx 0$ is the gauge-fixing for the superhamiltonian constraint, which is interpreted as an elliptic equation for the conformal factor of the 3-metric], for numerical solutions of Einstein equations and in the proof of the positive gravitational energy conjecture, let us give some results about these hypersurfaces [66,107] [see Ref. [71] for solutions of the Einstein's equations with matter, which do not admit constant mean extrinsic curvature hypersurfaces].

In Refs. [108,55,65] it is shown that ${}^3K$ defines the chosen rate of volume expansion of an initial slice Σ_τ relative to local proper time. In fact, an element of proper volume $\sqrt{\gamma} d^3\sigma$ [$\gamma = {}^3g$] on a spacelike hypersurface Σ_τ undergoes in the next unit interval of proper time, as measured normal to Σ_τ, a fractional increase of proper volume given by

$$-{}^3K = l^\mu{}_{;\mu} = -{}^3g^{\mu\nu}\, {}^3K_{\mu\nu} = \frac{1}{2}{}^3g^{\mu\nu} \mathcal{L}_l\, {}^3g_{\mu\nu} = \frac{1}{\sqrt{\gamma}} \mathcal{L}_l \sqrt{\gamma} = \mathcal{L}_l (ln \sqrt{\gamma}) = -\Theta({}^4g, l),$$

where Θ is the "expansion or dilatation" of l^μ [so that ${}^3K$ is equal to the "convergence of the normals l^μ" in $(M^4, {}^4g)$; see Appendix A of I]. For the volume to be extremal this quantity must vanish at every point of Σ_τ [this is satisfied in a Friedmann closed universe and in a Taub closed universe at that value of the natural time-coordinate t at which the universe switches from expansion to recontraction, so that the sign of ${}^3K$ could be used to distinguish the expansion and contraction epochs; see also Refs. [76]]. Moreover, it can be shown [65] that the rate of change of ${}^3K$ in timelike directions tends to be positive as a consequence of the equations of motion [for a freely falling observer (${}^3a^\mu = 0$) one has $\mathcal{L}_l\, {}^3K \geq 0$, i.e. ${}^3K$ increases with respect to the local standard of proper time; ${}^3K$ is essentially the volume Hubble parameter], and that ${}^3K$ defines a definite class of foliations. See Ref. [59] for York cosmic time $\mathcal{T}$ versus proper time.

Instead, Misner's choice [53] of the internal intrinsic time $\Omega = -\frac{1}{3} ln \sqrt{\gamma} = -q = -2 ln\, \phi$ [the logarithm of the volume] is acceptable for open always expanding universes; for closed universes, Ω stops its forward flow at a moment of maximum expansion and begins to run backward. The other problem with Ω is that it is "not a scalar" and thus has utility only in the presence of a definite choice of 3-dimensional coordinates. Moreover, Ω contains the conformal factor of the 3-metric, which is the natural variable in which to solve the superhamiltonian constraint. Since $\mathcal{L}_l\, \Omega = \kappa\, {}^3K$, both these variables define the same family of hypersurfaces in homogeneous models. The use of ${}^3K$ as time (and of the associated foliations) does not depend on any assumptions of homogeneity, nor does it restrict in any way the anisotropy of the universe. Finally, with the choice of ${}^3K$ as "time", its conjugate variable [the natural Hamiltonian to be introduced as an "energy" after having solved the constraints in this approach] is the scale factor $\sqrt{\gamma}$, so that the "energy" becomes equal to the volume of the universe. See Refs. [76] for the connection of ${}^3K$ with the "many-fingered time approach".

A number of theorems regarding the mean extrinsic curvature exist, based on the linearization of the map $\mathcal{T}$.

a) Compact Slices- See the results in Ref. [109]

b) Noncompact Slices. More difficult to study because of a lack of sufficient knowledge of the invertibility of the corresponding Laplace operator. There is one important particular case: Σ_τ diffeomorphic to R^3 with ${}^4g$ asymptotic to the Minkowski 4-metric at spatial infinity. Theorem [109,110]: Every Lorentzian manifold $(M^4, {}^4g)$ in a neighbourhood of Minkowski spacetime $(R^4, {}^4\eta)$ admits a "maximal" (i.e. with $\mathcal{T} = Tr\,{}^3K = 0$) spacelike submanifold. It has been conjectured that all spacetimes satisfying the strong energy condition, and that can be continuously deformed into Minkowski spacetime, admit a maximal hypersurface. It is known that there exists a maximal submanifold of Minkowski spacetime passing through a bounded regularly spacelike 2-dimensional boundary [see Ref. [66] for references]. One knows also that "maximal slicing" exists in many cases when Σ_τ is "noncompact" but "not" diffeomorphic to R^3, for example, in the Schwarzschild, Reissner-Nordström and Kerr spacetimes as well as general vacuum static and stationary spacetimes [66,111].

One may use $\mathcal{T} = -\frac{4}{3}\epsilon^3K = const. \neq 0$ slices when Σ_τ is "noncompact"; here the slices are analogous to the "mass hyperboloids" of Minkowski spacetime and are asymptotically null [112,111]. In Minkowski spacetime $(R^4, {}^4\eta)$ the most interesting spacelike slices [55] are the (future and past) "mass hyperboloids" [asymptotic to the (future and past) null cones] with ${}^3K = const.$ and the standard $x^o = const.$ hyperplanes with ${}^3K = 0$.

In Ref. [66] it is reported that when, in the asymptotically flat case, one uses weighted Sobolev spaces, which prevent the existence of stability subgroups of gauge transformations (and therefore the Gribov ambiguity) for the spin connection, then there are no "conformal Killing vectors" for the Riemann manifold $(\Sigma_\tau, {}^3g)$ [the equation $\xi^u\partial_u\, {}^3g_{rs} + \partial_r\xi^u\, {}^3g_{us} + \partial_s\xi^u\, {}^3g_{ru} = \frac{1}{3}{}^3g_{rs}[\xi^u\, {}^3g^{mn}\partial_u\, {}^3g_{mn} + 2\partial_u\xi^u]$ has no solution $\xi^u\partial_u$]. In these weighted Sobolev spaces one can also show [66,113] that the Laplace-Beltrami operator $\triangle_{{}^3g}$ is an isomorphism: this implies that $(\Sigma_\tau, {}^3g)$ has no isometries [i.e. Killing vectors $\xi^u\partial_u$ satisfying $\xi^u\partial_u\, {}^3g_{rs} + \partial_r\xi^u\, {}^3g_{us} + \partial_s\xi^u\, {}^3g_{ru} = 0$], because no such vector can tend to zero to infinity.

B) Let us now consider two kinds of decompositions of symmetric 3-tensors $T^{rs} = T^{sr} = {}^3T^{rs}$ defined on a Riemannian 3-manifold $(\Sigma_\tau, {}^3g_{rs})$, whose validity in the noncompact case requires the use of weighted Sobolev spaces. A closed manifold means a compact manifold without boundary. One can show that in closed manifolds Σ_τ every vector field on Σ_τ is the sum of a Killing vector field for ${}^3g$ and the divergence of a symmetric 3-tensor field [all vector fields can be written globally on Σ_τ as such divergences if and only if ${}^3g$ has no Killing field].

1) "Transverse decomposition" [114]. Following Ref. [108], it is defined as

$$
\begin{aligned}
T^{rs} &= T^{rs}_t + T^{rs}_l, \\
\\
T^{rs}_l &= (KX)^{rs} \equiv {}^3\nabla^r X^s + {}^3\nabla^s X^r = (KX)^{sr}, \\
{}^3\nabla_s T^{rs}_t &= T^{rs}{}_{|s} = {}^3\nabla_s T^{rs} - {}^3\nabla_s (KX)^{rs} = 0, \\
&\Rightarrow \\
{}^3\nabla_s (KX)^{rs} &\equiv ({}^3\triangle_K X)^r = {}^3\triangle X^r + {}^3\nabla^r({}^3\nabla_s X^s) + {}^3R^r{}_s X^s = {}^3\nabla_s T^{rs}.
\end{aligned}
\tag{C1}
$$

Here the longitudinal part $T_l^{rs} = (KX)^{rs}$ is the "Killing form" of X^r [$(KX)_{rs} = \mathcal{L}_X\, {}^3g_{rs}$ and $(KX)_{rs} = 0$ is the Killing equation for determining the infinitesimal isometries of the 3-metric ${}^3g_{rs}$]. While ${}^3\triangle$ is the ordinary Laplacian for the 3-metric, ${}^3\triangle_K$ is a linear second-order vector operator, the K-Laplacian.

The trace-free part of T_t^{rs}, i.e. $T_t^{rs} - \frac{1}{3}\, {}^3g^{rs} T_t$ [$T_t = {}^3g_{rs} T_t^{rs} = T - 2\, {}^3\nabla_r X^r$, $T = {}^3g_{rs} T^{rs}$], is no longer transverse, because, in general, ${}^3\nabla_r T_t = {}^3\nabla_r (T - 2\, {}^3\nabla_s X^s) \neq 0$.

If we make the transverse decomposition of this trace-free part of T^{rs}, we get

$$T^{rs} - \frac{1}{3}\, {}^3g^{rs} T = (T^{rs} - \frac{1}{3}\, {}^3g^{rs} T)_t + (KS)^{rs}$$

for some S^r. Now, $(T^{rs} - \frac{1}{3}\, {}^3g^{rs} T)_t$ is transverse, but not trace-free: ${}^3g_{rs} (T^{rs} - \frac{1}{3}\, {}^3g^{rs} T)_t = 2\, {}^3\nabla_s S^s \neq 0$ in general.

2) "Transverse-Traceless decomposition" [108] [see Ref. [115] for more mathematical properties on the solution of the elliptic equation for Y^r, which is connected to the linearization of the Cotton-York tensor]. It is defined as

$$\begin{aligned}
T^{rs} &= T^{rs}_{TT} + T^{rs}_L + T^{rs}_{Tr},\\
T^{rs}_{Tr} &= \frac{1}{3}{}^3g^{rs} T, \qquad T = {}^3g_{rs} T^{rs},\\
{}^3g_{rs} T^{rs}_{TT} &= {}^3\nabla_s\, T^{rs}_{TT} = 0,\\
T^{rs}_L &= (LY)^{rs} \equiv {}^3\nabla^r Y^s + {}^3\nabla^s Y^r - \frac{2}{3}{}^3g^{rs}\, {}^3\nabla_u Y^u =\\
&= (KY)^{rs} - \frac{2}{3}{}^3g^{rs}\, {}^3\nabla_u Y^u,\\
{}^3g_{rs} T^{rs}_L &= 0,\\
{}^3\nabla_s (LY)^{rs} &\equiv ({}^3\triangle_L Y)^r = {}^3\triangle Y^r + \frac{1}{3}{}^3\nabla^r ({}^3\nabla_s Y^s) + {}^3R^r{}_s Y^s =\\
&= {}^3\nabla_s (T^{rs} - \frac{1}{3}{}^3g^{rs} T).
\end{aligned} \tag{C2}$$

Now, $(LY)^{rs}$ is the "conformal Killing form" of Y^r, because, if ${}^3\tilde{g}_{rs} = \gamma^{-1/3}\, {}^3g_{rs}$ is the "conformal metric" (independent of arbitrary overall scale changes: if ${}^3g_{rs} \mapsto \phi\, {}^3g_{rs}$ then ${}^3\tilde{g}_{rs} \mapsto {}^3\tilde{g}_{rs}$), then $\mathcal{L}_Y\, {}^3\tilde{g}_{rs} = \gamma^{-1/3}\, (LY)_{rs}$ (this describes the action of infinitesimal coordinate transformations on the conformal metric) and $\mathcal{L}_Y\, {}^3\tilde{g}_{rs} = 0$ is the conformal Killing equation, determining the conformal Killing vectors (if any) of ${}^3\tilde{g}_{rs}$.

The TT-decomposition gives a unique result. It turns out that Tr-, TT- and L-tensors are mutually orthogonal. This is the content of York's splitting theorem [66].

It can be shown [108] that one has

$$\begin{aligned}
T^{rs}_{TT} &= (T^{rs}_t)_{TT} = (T^{rs}_{TT})_t,\\
(T^{rs}_{TT})_l &= (KV)^{rs} = 0,\\
(T^{rs}_l)_{TT} &= 0,\\
(T^{rs}_t)_L &= (LM)^{rs} = (LY)^{rs} - (LX)^{rs} =
\end{aligned}$$

$$
\begin{aligned}
&= T^{rs}_L - [(KX)^{rs} - \frac{2}{3}{}^3g^{rs}\,{}^3\nabla_u X^u] = T^{rs}_L - T^{rs}_l + \frac{1}{3}{}^3g^{rs}T_l,\\
T^{rs}_t &= (T^{rs}_t)_{TT} + (T^{rs}_t)_L + (T^{rs}_t)_{Tr} =\\
&= T^{rs}_{TT} + [L(Y-X)]^{rs} + \frac{1}{3}{}^3g^{rs}T_l =\\
&= T^{rs}_{TT} + T^{rs}_L - T^{rs}_l + \frac{1}{3}{}^3g^{rs}T_l,\\
T^{rs}_l &= (T^{rs}_l)_{TT} + (T^{rs}_l)_L + (T^{rs}_l)_{Tr} =\\
&= (LZ)^{rs} + \frac{1}{3}{}^3g^{rs}T_l, \quad for \quad some \quad Z^r,\\
T &= T_l + T_t, \qquad T_l = 2\,{}^3\nabla_r X^r,\\
\\
T^{rs}_{TT} &= (T^{rs}_{TT})_t + (T^{rs}_{TT})_l = (T^{rs}_{TT})_t,\\
T^{rs}_L &= (T^{rs}_L)_t + (T^{rs}_L)_l = (T^{rs}_t)_L + T^{rs}_l - \frac{1}{3}{}^3g^{rs}T_l. \qquad \text{(C3)}
\end{aligned}
$$

For closed Σ_τ one has the theorem [108]: Let Y^r be a harmonic function of ${}^3\triangle_L$ with nowhere vanishing norm on a closed manifold M; then, there always exists a manifold $\tilde{M}$ conformally related to M for which Y^r is a harmonic function of ${}^3\tilde{\triangle}_K$.

Every transverse symmetric tensor T^{rs}_t on $(\Sigma_\tau, {}^3g_{rs})$ can be split uniquely and orthogonally into a sum of a "transverse tensor with vanishing trace" and a "transverse tensor with nonvanishing trace". From $(T^{rs}_t)_{TT} = T^{rs}_{TT}$ and $T^{rs}_t = T^{rs}_{TT} + (LM)^{rs} + \frac{1}{3}{}^3g^{rs}T_t$ with $M^r = Y^r - X^r$, we get

$$
\begin{aligned}
T^{rs}_t &= T^{rs}_{TT} + T^{rs}_{Tr,t},\\
\\
T^{rs}_{Tr,t} &= (LM)^{rs} + \frac{1}{3}{}^3g^{rs}T_t,\\
{}^3\nabla_s T^{rs}_{Tr,t} &= {}^3\nabla_s (LM)^{rs} + \frac{1}{3}{}^3\nabla^r\, T_t = {}^3\nabla_s (T^{rs}_t - \frac{1}{3}{}^3g^{rs}T_t) + \frac{1}{3}{}^3\nabla^r T_t = 0,\\
{}^3g_{rs}T^{rs}_{Tr,t} &= {}^3g_{rs}[(LM)^{rs} + \frac{1}{3}{}^3g^{rs}T_t] = T_t. \qquad \text{(C4)}
\end{aligned}
$$

It follows that the gradient of the trace of a transverse tensor is always globally orthogonal to conformal Killing vectors, when Σ_τ is closed.

Therefore, T^{rs}_t contains a TT part, T^{rs}_{TT}, plus another tensor $T^{rs}_{Tr,t}$ which can be expressed as a functional only of T_t due to the equations ${}^3\nabla_s (LM)^{rs} = -\frac{1}{3}{}^3\nabla^r T_t$. Since the supermomentum constraints for metric gravity, i.e. 3 of the Einstein equations, imply that

${}^3K^{rs} \circeq {}^3K^{rs}_t = {}^3K^{rs}_{TT} + {}^3K^{rs}_{Tr,t}$ with ${}^3K^{rs}_{Tr,t} = (LM)^{rs} + \frac{1}{3}{}^3g^{rs}\,{}^3K$,

one can say that ${}^3K^{rs}_{TT}$ contains the "wave" part [purely gravitational spin-two TT-tensor], while ${}^3K = Tr\,{}^3K$ is a kinematical function defining an essentially arbitrary "gauge" degree of freedom.

C) In the conformal approach it is assumed that the superhamiltonian constraint becomes the "scale or Lichnerowicz equation" [66], which is a quasilinear elliptic equation for

$$\phi(\tau,\vec{\sigma}) = e^{q(\tau,\vec{\sigma})} = [\gamma(\tau,\vec{\sigma})]^{1/12} = {}^3e^{1/6} = [-\mathcal{P}_{\mathcal{T}}(\tau,\vec{\sigma})]^{1/6}.$$

In Ref. [66] it is shown that ϕ and a 3-vector X^r (the vector part of the TT-decomposition) can be interpreted physically as generalizations of the single potential function that satisfies Poisson's equation in Newtonian gravity: one has for $r = |\vec{\sigma}| \rightarrow \infty$ the following results

$$\phi = 1 + E/16\pi r + ...,$$
$$X^r = \frac{1}{32\pi r^3} P^s(7r^2\delta^r_s + \sigma^r\sigma^s) + ...,$$

where E and P^r are the asymptotic Poincaré translation charges [modulo the supertranslations connected to asymptotic gauge transformations].

The conformal approach can be formulated either in terms of ${}^3g_{rs}$, ${}^3K_{rs}$ or in terms of the ADM canonical variables ${}^3g_{rs}$, ${}^3\tilde{\Pi}^{rs}$ (see Section V of I). The ADM supermomentum constraints for the tensor density ${}^3\tilde{\Pi}^{rs}$ are proportional to the Einstein equations

$${}^4\bar{G}_{lr} = \epsilon[\sqrt{\gamma}({}^3K_r{}^s - \delta^s_r\,{}^3K)]_{|s} \circeq 0\,, \text{ i.e.}$$
$$0 \approx {}^3\tilde{\Pi}^{rs}_{|s} = \epsilon k[\sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\,{}^3K)]_{|s} = \epsilon k\sqrt{\gamma}\,{}^4\bar{G}_l{}^r \circeq 0.$$

One has ${}^3\tilde{\Pi} = -2\epsilon k\sqrt{\gamma}\,{}^3K$ with ${}^3K = {}^3g^{rs}\,{}^3K_{rs} = Tr\,{}^3K$.

The idea is to make a suitable separation between physical and unphysical degrees of freedom to identify four candidates for the variables in which either the Einstein equations ${}^4\bar{G}_{ll} \circeq 0$, ${}^4\bar{G}_{lr} \circeq 0$, or the ADM constraints, have to be solved [in the ADM approach a separation is made based on a TT-decomposition referred to a flat background spacetime; this is suited for weak fields and linearized theory].

First of all, one makes a "conformal transformation" on the 3-metric,

$${}^3g_{rs} = \phi^4\,{}^3\check{g}_{rs}$$

[${}^3g^{rs} = \phi^{-4}\,{}^3\check{g}^{rs}$; ${}^3R = \phi^{-4}\,{}^3\check{R} - 8\phi^{-5}\,{}^3\check{\triangle}\phi$ with ${}^3\check{\triangle}$ the Laplacian for the 3-metric ${}^3\check{g}_{rs}$] with ϕ an arbitrary definite positive function (positivity is crucial for the study of the existence and uniqueness of solutions of the Lichnerowicz equation [66]); essentially, one uses

$$\phi = e^{q/2} = \gamma^{1/12} = {}^3e^{1/6},$$

so that ${}^3\check{g}_{rs} = {}^3\tilde{g}_{rs}$, where ${}^3\tilde{g}_{rs}$ is the "conformal metric" with $det\,{}^3\tilde{g}_{rs} = 1$ [at each point it gives only the ratio between any two "local" distances; the absolute distances are fixed by the scale factor ϕ].

Secondly, one defines the trace-free part ${}^3A^{rs}$ (also called the "distortion tensor"), ${}^3g_{rs}\,{}^3A^{rs} = 0$, of ${}^3K^{rs}$:

$${}^3K^{rs} = {}^3A^{rs} + \frac{1}{3}\,{}^3g^{rs}\,{}^3K \text{ or}$$
$${}^3\tilde{\Pi}^{rs} = \epsilon k\sqrt{\gamma}({}^3A^{rs} - \frac{2}{3}\,{}^3g^{rs}\,{}^3K) = {}^3\tilde{\Pi}^{rs}_A + \frac{1}{3}\,{}^3g^{rs}\,{}^3\tilde{\Pi},\ {}^3g_{rs}\,{}^3\tilde{\Pi}^{rs}_A = 0.$$

Now the supermomentum constraints become

either ${}^3A^{rs}{}_{|s} - \frac{2}{3}\,{}^3\nabla^r\,{}^3K \circeq 0$ or ${}^3\tilde{\Pi}^{rs}_{A\,|s} + \frac{1}{3}\,{}^3\nabla^r\,{}^3\tilde{\Pi} \approx 0$.

Then, in the simplest version of the approach [66,55], one makes the "conformal rescaling" ${}^3A^{rs} = \phi^{-10}\,{}^3\check{A}^{rs}$ $[{}^3\tilde{\Pi}^{rs}_A = \phi^{-10}\,{}^3\check{\tilde{\Pi}}^{rs}_A]$, so that

${}^3A^{rs}{}_{|s} = {}^3\nabla_s\,{}^3A^{rs} = \phi^{-10}\,{}^3\check{\nabla}_s\,{}^3\check{A}^{rs}$, ${}^3\nabla^r\,{}^3K = \phi^{-4}\,{}^3\check{\nabla}^r\,{}^3K$

$[{}^3\nabla_s\,{}^3\tilde{\Pi}^{rs}_A = \phi^{-10}\,{}^3\check{\nabla}_s\,{}^3\check{\tilde{\Pi}}^{rs}_A$, ${}^3\nabla^r\,{}^3\tilde{\Pi} = \phi^{-4}\,{}^3\check{\nabla}^r\,{}^3\tilde{\Pi}]$.

The next step is to make a TT-decomposition of ${}^3\check{A}^{rs}$ $[{}^3\check{\tilde{\Pi}}^{rs}_A]$: ${}^3\check{A}^{rs} = {}^3\check{A}^{rs}_{TT} + {}^3\check{A}^{rs}_L$ with ${}^3\check{A}^{rs}_L = (L\check{W})^{rs}$ and

${}^3\check{\nabla}_s\,{}^3\check{A}^{rs}_L = ({}^3\check{\triangle}_L\check{W})^r = {}^3\check{\nabla}_s\,{}^3\check{A}^{rs} \circeq \frac{2}{3}\phi^6\,{}^3\check{\nabla}^r\,{}^3K$

$[{}^3\check{\tilde{\Pi}}^{rs}_A = {}^3\check{\tilde{\Pi}}^{rs}_{A,TT} + {}^3\check{\tilde{\Pi}}^{rs}_{A,L}$ with ${}^3\check{\tilde{\Pi}}^{rs}_{A,L} = (L\check{W}_\pi)^{rs}$ and ${}^3\check{\nabla}_s\,{}^3\check{\tilde{\Pi}}^{rs}_{A,L} = ({}^3\check{\triangle}_L\check{W}_\pi)^r = {}^3\check{\nabla}_s\,{}^3\check{\tilde{\Pi}}^{rs}_A \approx -\frac{1}{3}\phi^6\,{}^3\check{\nabla}^r\,{}^3\tilde{\Pi}]$. The equation ${}^4\bar{G}_{ll} \circeq 0$ becomes the scale or Lichnerowicz equation

$$8\,{}^3\check{\triangle}\phi - {}^3\check{R}\phi + ({}^3\check{A}^{rs}_{TT} + {}^3\check{A}^{rs}_L)({}^3\check{A}_{TT,rs} + {}^3\check{A}_{L,rs})\phi^{-7} - \frac{2}{3}({}^3K)^2\phi^5 \circeq 0, \qquad \text{(C5)}$$

[the same happens for the superhamiltonian constraint], which is in general coupled with the equations ${}^4\bar{G}_{lr} \circeq 0$

$$ {}^3\check{\nabla}_s\,{}^3\check{A}^{rs}_L = ({}^3\check{\triangle}_L\check{W})^r = {}^3\check{\nabla}_s\,{}^3\check{A}^{rs} \circeq \frac{2}{3}\phi^6\,{}^3\check{\nabla}^r\,{}^3K. \qquad \text{(C6)}$$

The four Einstein equations [ADM constraints] have to be solved in ϕ [the "conformal or scale factor"] and in the longitudinal part ${}^3\check{A}^{rs}_L$ $[{}^4\check{\tilde{\Pi}}^{rs}_{A,L}]$ of ${}^3\check{A}^{rs}$ $[{}^3\check{\tilde{\Pi}}^{rs}_A]$, i.e. in the vector $\check{W}^r$ $[\check{W}^r_\pi]$ [see the previous subsection B)], which is named the "gravitomagnetic vector potential"; these four functions become functionals of ${}^3\check{g}_{rs}$ and of the parts ${}^3\check{A}^{rs}_{TT}$ and ${}^3K$ of ${}^3K_{rs}$ $[{}^3\check{\tilde{\Pi}}^{rs}_{A,TT}$ and ${}^3\tilde{\Pi}$ of ${}^3\tilde{\Pi}^{rs}]$. While ${}^3K$ $[{}^3\tilde{\Pi}]$, the mean extrinsic curvature, is interpreted as the internal extrinsic time conjugated to the momentum ϕ, 3 components of ${}^3\check{g}_{rs}$ have to be interpreted as conjugate to the vector $\check{W}^r$; ${}^3\check{A}^{rs}_{TT}$ [the free part of the conformally rescaled "distorsion tensor"] and the remaining two degrees of freedom hidden in ${}^3\check{g}_{rs}$ are the genuine (gravitational wave) physical degrees of freedom in this reduction. For constant or vanishing (maximal slicing) ${}^3K$, the supermomentum constraints decouple from the Lichnerowicz equation. See Ref. [75] for a review on the existence and unicity of the solutions of Eqs.(C5), (C6) when ${}^3K = 0$ *or const.* and Refs. [66,68,69] for the classification of the known solutions of this equation. The Yamabe theorem is a fundamental tool in this classification [116].

In presence of matter in a closed universe, the conformal current ${}^3j^r$ of mass-energy has to be orthogonal to the conformal Killing vectors of the conformal 3-metric (if any). This is the "condition of confinability" for the gravitomagnetic vector potential $\check{W}^r$ [75,117] [it is like in electrostatic, where, in a closed space, the Poisson equation $\triangle\phi = -4\pi\rho$ implies $\int d^3x\sqrt{\gamma}\rho = 0$ (i.e. the vanishing of the total source charge) [75]].

The previous decomposition suggests to use the variables $\mathcal{T} = -\frac{4}{3}\epsilon k\, {}^3K = \frac{2}{3\sqrt{\gamma}}\, {}^3\tilde{\Pi}$, $\mathcal{P}_{\mathcal{T}} = -\sqrt{\gamma}$, ${}^3\sigma_{rs} = {}^3g_{rs}/\gamma^{1/3}$, ${}^3\tilde{\Pi}^{rs}_A = {}^3\tilde{\Pi}^{rs} - \frac{1}{3}{}^3g^{rs}\, {}^3\tilde{\Pi}$, which satisfy the Poisson brackets

$$
\begin{aligned}
&\{\mathcal{T}(\tau,\vec{\sigma}), \mathcal{P}_{\mathcal{T}}(\tau,\vec{\sigma}^{'}\} = -\delta^3(\vec{\sigma},\vec{\sigma}),\\
&\{{}^3\sigma_{rs}(\tau,\vec{\sigma}), {}^3\tilde{\Pi}^{uv}_A(\tau,\vec{\sigma}^{'})\} = [\frac{1}{2}(\delta^u_r\delta^v_s - \delta^v_r\delta^u_s) - \frac{1}{3}\, {}^3\sigma^{uv}\, {}^3\sigma_{rs}](\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}),\\
&\{{}^3\tilde{\Pi}^{rs}_A(\tau,\vec{\sigma}), {}^3\tilde{\Pi}^{uv}_A(\tau,\vec{\sigma}^{'})\} = \frac{1}{3}({}^3\sigma^{uv}\, {}^3\tilde{\Pi}^{rs}_A - {}^3\sigma^{rs}\, {}^3\tilde{\Pi}^{uv}_A)(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}).
\end{aligned}
\tag{C7}
$$

In Ref. [67] it is shown that there exists a canonical basis $[{}^3\sigma_{rs}, {}^3\tilde{\Pi}^{rs}_{TT}]$ hidden in the variables ${}^3\sigma_{rs}$, ${}^3\tilde{\Pi}^{rs}_A$ [but it has never been found explicitly] and that one can define the reduced phase space (the conformal superspace) $\tilde{\mathcal{S}}$, in which one has gone to the quotient with respect to the space diffeomorphisms and to the conformal rescalings. It is also shown that one can define a "York map" from this reduced phase space to the subset of the standard phase superspace (quotient of the ADM phase space with respect to the space diffeomorphisms plus the gauge transformations generated by the superhamiltonian constraint; it is the phase space of the superspace, the configuration space obtained from the 3-metrics going to the quotient with respect to the space- and 'time'- diffeomorphisms of the ADM formalism) defined by the condition ${}^3K = const.$.

The "conformal superspace" $\tilde{\mathcal{S}}$ may be defined as the space of conformal 3-geometries on "closed" manifolds and can be identified in a natural way with the space of conformal 3-metrics modulo space diffeomorphisms, or, equivalently, with the space of Riemannian 3-metrics modulo space diffeomorphisms and conformal transformations of the form ${}^3g_{rs} \mapsto \phi^4\, {}^3g_{rs}$, $\phi > 0$. Instead, the "ordinary superspace" $\mathcal{S}$ is the space of Lorentzian 4-metrics modulo spacetime diffeomorphisms. In this way a bridge is built towards the phase superspace, which is mathematically connected with the Moncrief splitting theorem [118,66] valid for closed Σ_τ [see however Ref. [66] for what is known in the asymptotically flat case by using weighted Sobolev spaces].

APPENDIX D: 3-TENSORS IN THE FINAL CANONICAL BASIS.

By using the definitions given in I, Eqs.(102) imply the following expressions for the field strengths and curvature tensors of $(\Sigma_\tau, {}^3g)$ [we also give their limits for $r_{\bar a} \rightarrow 0$ and $q \rightarrow 0$]

$$\begin{aligned}
{}^3\hat{\Omega}_{rs(a)} &= \epsilon_{(a)(b)(c)} \sum_u \delta_{(c)u} \\
&\Big(\delta_{(b)s} e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a s}-\gamma_{\bar a u})r_{\bar a}} \Big[\frac{1}{\sqrt{3}}(\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b s}\partial_u r_{\bar b}) \sum_{\bar c}(\gamma_{\bar c s}-\gamma_{\bar c u})\partial_r r_{\bar c} + \\
&+\partial_u\partial_r q + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b s}\partial_u\partial_r r_{\bar b}\Big] - \\
&-\delta_{(b)r} e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a r}-\gamma_{\bar a u})r_{\bar a}} \Big[\frac{1}{\sqrt{3}}(\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_u r_{\bar b}) \sum_{\bar c}(\gamma_{\bar c r}-\gamma_{\bar c u})\partial_s r_{\bar c} + \\
&+\partial_u\partial_s q + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_u\partial_s r_{\bar b}\Big]\Big) + \\
&+\frac{1}{2}\sum_{uv}\Big[\delta_{(a)(b)}\epsilon_{(c)(d)(e)} - \delta_{(a)(c)}\epsilon_{(b)(d)(e)} + \delta_{(a)(d)}\epsilon_{(e)(c)(b)} - \delta_{(a)(e)}\epsilon_{(d)(c)(b)}\Big] \\
&e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a r}+\gamma_{\bar a s}-\gamma_{\bar a u}-\gamma_{\bar a v})r_{\bar a}} \Big(\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_u r_{\bar b}\Big)\Big(\partial_v q + \frac{1}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c s}\partial_v r_{\bar c}\Big) \\
&\rightarrow_{r_{\bar a}\rightarrow 0} \epsilon_{(a)(b)(c)}\sum_u \delta_{(c)u}\Big[\delta_{(b)s}\partial_u\partial_r q - \delta_{(b)r}\partial_u\partial_s q\Big] + \\
&+\frac{1}{2}\Big[\delta_{(a)(b)}\epsilon_{(c)(d)(e)} - \delta_{(a)(c)}\epsilon_{(b)(d)(e)} + \delta_{(a)(d)}\epsilon_{(e)(c)(b)} - \delta_{(a)(e)}\epsilon_{(d)(c)(b)}\Big]\partial_u q\partial_v q \rightarrow_{q\rightarrow 0} 0, \\
&\rightarrow_{q\rightarrow 0} \frac{1}{\sqrt{3}}\epsilon_{(a)(b)(c)}\sum_u \delta_{(c)u}\Big(\delta_{(b)s} e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a s}-\gamma_{\bar a u})r_{\bar a}}\, . \\
&\sum_{\bar b}\gamma_{\bar b s}\Big[\partial_r\partial_u r_{\bar b} + \frac{1}{\sqrt{3}}\partial_u r_{\bar b}\sum_{\bar c}(\gamma_{\bar c s}-\gamma_{\bar c u})\partial_r r_{\bar c}\Big] - \\
&-\delta_{(b)r} e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a r}-\gamma_{\bar a u})r_{\bar a}} \sum_{\bar b}\gamma_{\bar b r}\Big[\partial_s\partial_u r_{\bar b} + \frac{1}{\sqrt{3}}\partial_u r_{\bar b}\sum_{\bar c}(\gamma_{\bar c r}-\gamma_{\bar c u})\partial_s r_{\bar c}\Big]\Big) + \\
&+\frac{1}{6}\sum_{uv}\Big[\delta_{(a)(b)}\epsilon_{(c)(d)(e)} - \delta_{(a)(c)}\epsilon_{(b)(d)(e)} + \delta_{(a)(d)}\epsilon_{(e)(c)(b)} - \delta_{(a)(e)}\epsilon_{(d)(c)(b)}\Big] \\
&e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a r}+\gamma_{\bar a s}-\gamma_{\bar a u}+\gamma_{\bar a v})r_{\bar a}} \sum_{\bar b}\gamma_{\bar b r}\partial_u r_{\bar b}\sum_{\bar c}\gamma_{\bar c s}\partial_v r_{\bar c},
\end{aligned}$$

$$\begin{aligned}
{}^3\hat{R}_{rusv} &= (\delta_{rv}\delta_{su} - \delta_{rs}\delta_{uv}) e^{2(2q+\frac{1}{\sqrt{3}}\sum_{\bar c}(\gamma_{\bar c r}+\gamma_{\bar c u})r_{\bar c})} \\
&\sum_n \Big(\partial_n q + \frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}\partial_n r_{\bar a}\Big)\Big(\partial_n q + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_n r_{\bar b}\Big) + \\
&+e^{2(q+\frac{1}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c r}r_{\bar c})}\Big(\delta_{rv}\Big[\partial_s\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}\partial_s\partial_u r_{\bar a} + \\
&+\frac{1}{\sqrt{3}}\Big(\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}\partial_u r_{\bar a}\Big)\sum_{\bar b}(\gamma_{\bar b r}-\gamma_{\bar b u})\partial_s r_{\bar b} -
\end{aligned}$$

$$
\begin{aligned}
&-\Big(\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s}\partial_u r_{\bar a}\Big)\Big(\partial_s q+\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_s r_{\bar b}\Big)\Big]-\delta_{rs}\Big[\partial_v\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar a r}\partial_v\partial_u r_{\bar a}+\\
&+\frac{1}{\sqrt{3}}\Big(\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}\partial_u r_{\bar a}\Big)\sum_{\bar b}(\gamma_{\bar b r}-\gamma_{\bar b u})\partial_v r_{\bar b}-\\
&-\Big(\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v}\partial_u r_{\bar a}\Big)\Big(\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_v r_{\bar b}\Big)\Big]\Big)+\\
&+e^{2(q+\frac{1}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c u}r_{\bar c})}\Big(\delta_{su}\Big[\partial_v\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_v\partial_r r_{\bar a}+\frac{1}{\sqrt{3}}\Big(\partial_r q+\\
&+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_r r_{\bar a}\Big)\sum_{\bar b}(\gamma_{\bar b u}-\gamma_{\bar b r})\partial_v r_{\bar b}-\\
&-\Big(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}\partial_r r_{\bar a}\Big)\Big(\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_v r_{\bar b}\Big)\Big]-\\
-\ &\delta_{uv}\Big[\partial_s\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_s\partial_r r_{\bar a}+\\
&+\frac{1}{\sqrt{3}}\Big(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_r r_{\bar a}\Big)\sum_{\bar b}(\gamma_{\bar b u}-\gamma_{\bar b r})\partial_s r_{\bar b}-\\
&-\Big(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s}\partial_r r_{\bar a}\Big)\Big(\partial_s q+\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_s r_{\bar b}\Big)\Big]\Big)\\
&\to_{r_{\bar a}\to 0}(\delta_{rv}\delta_{su}-\delta_{rs}\delta_{uv})e^{4q}\sum_n(\partial_n q)^2+\\
&+e^{2q}\Big(\delta_{rv}[\partial_s\partial_u q-\partial_s q\partial_u q]-\delta_{rs}[\partial_v\partial_u q-\partial_v q\partial_u q]+\\
&+\delta_{su}[\partial_v\partial_r q-\partial_v q\partial_r q]-\delta_{uv}[\partial_s\partial_r q-\partial_s q\partial_r q]\Big)\to_{q\to 0}0,\\
&\to_{q\to 0}\frac{1}{3}(\delta_{rv}\delta_{su}-\delta_{rs}\delta_{uv})e^{\frac{2}{\sqrt{3}}\sum_{\bar c}(\gamma_{\bar c r}+\gamma_{\bar c u})r_{\bar c}}\sum_n\sum_{\bar a\bar b}\gamma_{\bar a r}\gamma_{\bar b u}\partial_n r_{\bar a}\partial_n r_{\bar b}+\\
&+\frac{1}{\sqrt{3}}e^{\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c r}r_{\bar c}}\sum_{\bar a}\gamma_{\bar a r}\Big(\delta_{rv}\Big[\partial_s\partial_u r_{\bar a}+\\
&+\frac{1}{\sqrt{3}}\sum_{\bar b}(\gamma_{\bar b r}-\gamma_{\bar b u})\partial_u r_{\bar a}\partial_s r_{\bar b}-\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b s}\partial_s r_{\bar a}\partial_u r_{\bar b}\Big]-\\
&-\delta_{rs}\Big[\partial_v\partial_u r_{\bar a}+\frac{1}{\sqrt{3}}\sum_{\bar b}(\gamma_{\bar b r}-\gamma_{barbu})\partial_u r_{\bar a}\partial_v r_{\bar b}-\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b v}\partial_v r_{\bar a}\partial_u r_{\bar b}\Big]\Big)+\\
&+\frac{1}{\sqrt{3}}e^{\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c u}r_{\bar c}}\sum_{\bar a}\gamma_{\bar a u}\Big(\delta_{su}\Big[\partial_v\partial_r r_{\bar a}+\\
&+\frac{1}{\sqrt{3}}\sum_{\bar b}(\gamma_{\bar b u}-\gamma_{\bar b r})\partial_r r_{\bar a}\partial_v r_{\bar b}-\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b v}\partial_v r_{\bar a}\partial_r r_{\bar b}\Big]-\\
&-\delta_{uv}\Big[\partial_s\partial_r r_{\bar a}+\frac{1}{\sqrt{3}}\sum_{\bar b}(\gamma_{\bar b u}-\gamma_{\bar b r})\partial_r r_{\bar a}\partial_s r_{\bar b}-\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b s}\partial_s r_{\bar a}\partial_r r_{\bar b}\Big]\Big),
\end{aligned}
$$

$$
{}^3\hat R_{uv}=-\partial_u\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a u}+\gamma_{\bar a v})\partial_u\partial_v r_{\bar a}+
$$

$$
\begin{aligned}
&+\frac{1}{2}\Big[\Big(\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v}\partial_u r_{\bar a}\Big)\Big(\partial_v q-\frac{2}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_v r_{\bar b}\Big)+\\
&+\Big(\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar u}\partial_v r_{\bar a}\Big)\Big(\partial_u q-\frac{2}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b v} r_{\bar b}\Big)\Big]-\\
&-\frac{1}{2\sqrt{3}}\sum_n\Big[\Big(\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}\partial_y r_{\bar a}\Big)\sum_{\bar b}(\gamma_{\bar b n}-\gamma_{barbu})\partial_v r_{\bar b}+\\
&+\Big(\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}\partial_v r_{\bar a}\Big)\sum_{\bar b}(\gamma_{\bar b n}-\gamma_{\bar b v})\partial_u r_{\bar b}\Big]-\delta_{uv}e^{2(q+\frac{1}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c u}r_{\bar c})}\\
&\sum_n\Big((\partial_n q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_n r_{\bar a})(2\partial_n q-\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_n r_{\bar b})+\\
&+e^{-2(q+\frac{1}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c n}r_{\bar c})}\Big[\partial_n^2 q+\\
&+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_n^2 r_{\bar a}+\frac{1}{\sqrt{3}}(\partial_n q+\frac{1}{\sqrt{3}}\sum_{\bar a u}\partial_n r_{\bar a})\sum_{\bar b}(\gamma_{\bar b u}-2\gamma_{\bar b n})\partial_n r_{\bar b}\Big]\Big)\\
&\rightarrow_{r_{\bar a}\rightarrow 0}-\partial_u\partial_v q+\partial_u q\partial_v q-\delta_{uv}e^{2q}\sum_n\Big[2e^{2q}(\partial_n q)^2+\partial_n^2 q-(\partial_n q)^2\Big]\rightarrow_{q\rightarrow 0}0,\\
&\rightarrow_{q\rightarrow 0}\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a u}+\gamma_{\bar a v})\partial_u\partial_v r_{\bar a}-\frac{2}{3}\sum_{\bar a\bar b}\gamma_{\bar a u}\gamma_{\bar b v}\partial_v r_{\bar a}\partial_u r_{\bar b}-\\
&-\frac{1}{6}\sum_n\sum_{\bar a\bar b}\gamma_{\bar a n}\Big[(\gamma_{\bar b r}-\gamma_{\bar b u})\partial_u r_{\bar a}\partial_v r_{\bar b}+(\gamma_{\bar b n}-\gamma_{\bar b v})\partial_v r_{\bar a}\partial_u r_{\bar b}\Big]+\\
&+\frac{1}{\sqrt{3}}\delta_{uv}e^{\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c u}r_{\bar c}}\sum_n\Big(\frac{1}{\sqrt{3}}\sum_{\bar a\bar b}\gamma_{\bar a u}\gamma_{\bar b u}\partial_n r_{\bar a}\partial_n r_{\bar b}-\\
&-e^{-\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c n}r_{\bar c}}\sum_{\bar a}\gamma_{\bar a u}\Big[\partial_n^2 r_{\bar a}+\frac{1}{\sqrt{3}}\sum_{\bar b}(\gamma_{\bar b u}-2\gamma_{\bar b n})\partial_n r_{\bar a}\partial_n r_{\bar b}\Big]\Big),
\end{aligned}
$$

$$
\begin{aligned}
{}^3\hat R=&-\sum_{uv}\Big((\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_v r_{\bar a})(2\partial_v q-\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_v r_{\bar b})+\\
&+e^{-2(q+\frac{1}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c v}r_{\bar c})}\Big[\partial_v^2 q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_v^2 r_{\bar a}+\\
&+\frac{2}{\sqrt{3}}(\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_v r_{\bar a})\sum_{\bar b}(\gamma_{\bar b u}-\gamma_{\bar b v})\partial_v r_{\bar b}-\\
&-(\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v}\partial_v r_{\bar a})(\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_v r_{\bar b}\Big]\Big)+\\
&+\sum_u e^{-2(q+\frac{1}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c u}r_{\bar c})}\Big[-\partial_u^2 q+\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_u^2 r_{\bar a}+\\
&+(\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}\partial_u r_{\bar a})(\partial_u q-\frac{2}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_u r_{\bar b})\Big]\\
&\rightarrow_{r_{\bar a}\rightarrow 0}-6\sum_u(\partial_u q)^2-4e^{-2q}\sum_u\Big[\partial_u^2 q-(\partial_u q)^2\Big]\rightarrow_{q\rightarrow 0}0,
\end{aligned}
$$

$$
\begin{aligned}
&\rightarrow_{q\rightarrow 0}\; -\frac{1}{\sqrt{3}}\sum_{uv}\Big(-\frac{1}{\sqrt{3}}\sum_{\bar a\bar b}\gamma_{\bar au}\gamma_{\bar bu}\partial_v r_{\bar a}\partial_v r_{\bar b}+e^{-\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar cv}r_{\bar c}}\sum_{\bar a}\gamma_{\bar au}\cdot\\
&\Big[\partial_v^2 r_{\bar a}+\frac{2}{\sqrt{3}}\sum_{\bar b}(\gamma_{\bar bu}-\gamma_{\bar bv})\partial_v r_{\bar a}\partial_v r_{\bar b}-\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar bv}\partial_v r_{\bar a}\partial_v r_{\bar b}\Big]\Big)+\\
&+\frac{2}{\sqrt{3}}\sum_u e^{-\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar cu}t_{\bar c}}\sum_{\bar a}\gamma_{\bar au}\Big[\partial_u^2 r_{\bar a}+\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar bu}\partial_u r_{\bar a}\partial_u r_{\bar b}\Big].
\end{aligned}
\tag{D1}
$$

The Weyl-Schouten tensor

$$
{}^3C_{rsu}={}^3\nabla_u\,{}^3R_{rs}-{}^3\nabla_s\,{}^3R_{ru}-\tfrac{1}{4}({}^3g_{rs}\partial_u\,{}^3R-{}^3g_{ru}\partial_s\,{}^3R)
$$

of the 3-manifold $(\Sigma_\tau,{}^3g)$ [see after Eq.(9) of I and Ref. [106]] satisfies ${}^3C^r{}_{ru}=0$, ${}^3C_{rsu}=-{}^3C_{rus}$, ${}^3C_{rsu}+{}^3C_{urs}+{}^3C_{sur}=0$ and has 5 independent components. The related York's conformal tensor [65,63]

$$
{}^3Y^{rs}=\gamma^{1/3}\,\epsilon^{ruv}({}^3R_v{}^s-\tfrac{1}{4}\delta^s_v\,{}^3R)_{|u}=-\tfrac{1}{2}\gamma^{1/3}\epsilon^{ruv}\,{}^3g^{sm}\,{}^3C_{muv}
$$

[it is a tensor density of weight 5/3 and involves the third derivatives of the metric] is symmetric $[{}^3Y^{rs}={}^3Y^{sr}]$, traceless $[{}^3Y^r{}_r=0]$ and transverse $[{}^3Y^{rs}{}_{|s}=0]$ besides being invariant under 3-conformal transformations; therefore, it has only 2 independent components $[Y^{rs}=Y^{rs}_{TT}$ according to York's decomposition of Appendix C] and provides what York calls the pure spin-two representation of the 3-geometry intrinsic to Σ_τ. Its explicitly symmetric form is the Cotton-York tensor given by

$$
{}^3\mathcal{Y}^{rs}=\tfrac{1}{2}({}^3Y^{rs}+{}^3Y^{sr})=\tfrac{1}{2}\gamma^{1/3}(\epsilon^{ruv}\,{}^3g^{sc}+\epsilon^{suv}\,{}^3g^{rc})^3R_{vc|u}=-\tfrac{1}{4}\gamma^{1/3}(\epsilon^{ruv}\,{}^3g^{sm}+\epsilon^{suv}\,{}^3g^{rm})^3C_{muv}.
$$

A 3-manifold is conformally flat if and only if either the Weyl-Schouten or the Cotton-York tensor vanishes [63,65,106]. We have

$$
\begin{aligned}
{}^3C_{rsu}&={}^3\nabla_u\,{}^3R_{rs}-{}^3\nabla_s\,{}^3R_{ru}-\frac{1}{4}({}^3g_{rs}\partial_u\,{}^3R-{}^3g_{ru}\partial_s\,{}^3R)\mapsto\\
&\mapsto {}^3\hat C_{rsu}={}^3\hat R_{rs|u}-{}^3\hat R_{ru|s}-\frac{1}{4}e^{2q_r}(\delta_{rs}\partial_u\,{}^3\hat R-\delta_{ru}\partial_s\,{}^3\hat R),
\end{aligned}
$$

$$
\begin{aligned}
{}^3\mathcal{Y}_{mn}&=\frac{1}{2}\gamma^{1/3}\sum_{rsu}(\epsilon_{mur}\,{}^3g_{ns}+\epsilon_{nur}\,{}^3g_{ms})^3R_{rs|u}\mapsto\\
&\mapsto {}^3\hat{\mathcal{Y}}_{mn}=\frac{1}{2}e^{\frac{2}{3}\sum_v q_v}\sum_{rsu}e^{-2q_s}(\epsilon_{mur}\delta_{ns}+\epsilon_{nur}\delta_{ms})^3\hat R_{rs|u}=\\
&=\frac{1}{2}e^{2q}\sum_{rsu}e^{-2q_s}(\epsilon_{mur}\delta_{ns}+\epsilon_{nur}\delta_{ms})\cdot\\
&\Big(\partial_u\partial_r\partial_s(q_r+q_s-\sum_t q_t)-\partial_u(q_r+q_s)\partial_r\partial_s(q_r+q_s-\sum_t q_t)-\\
&-\partial_r q_u\partial_u\partial_s(q_u+q_s-\sum_t q_t)-\partial_s q_u\partial_u\partial_r(q_u+q_r-\sum_t q_t)-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\partial_u[\partial_r q_s \partial_s(2q_r - \sum_t q_t) + \partial_s q_r \partial_r(2q_s - \sum_t q_t)] - \\
&-\frac{1}{2}\sum_n \partial_u[\partial_r q_n \partial_s(q_n - q_r) + \partial_s q_n \partial_r(q_n - q_s)] + \\
&+\frac{1}{2}\partial_u(q_r + q_s)[\partial_r q_s \partial_s(2q_r - \sum_t q_t) + \partial_s q_r \partial_r(2q_s - \sum_t q_t)] + \\
&+\frac{1}{2}\partial_r q_u[\partial_u q_s \partial_s(2q_u - \sum_t q_t) + \partial_s q_u \partial_u(2q_s - \sum_t q_t)] + \\
&+\frac{1}{2}\partial_s q_u[\partial_u q_r \partial_r(2q_u - \sum_t q_t) + \partial_r q_u \partial_u(2q_r - \sum_t q_t)] + \\
&+\frac{1}{2}\partial_u(q_r + q_s)\sum_n[\partial_r q_n \partial_s(q_n - q_r) + \partial_s q_n \partial_r(q_n - q_s)] + \\
&+\frac{1}{2}\partial_r q_u \sum_n[\partial_u q_n \partial_s(q_n - q_u) + \partial_s q_n \partial_u(q_n - q_s)] + \\
&+\frac{1}{2}\partial_s q_u \sum_n[\partial_u q_n \partial_r(q_n - q_u) + \partial_r q_n \partial_u(q_n - q_r)] + \\
&+\delta_{rs}\, e^{2q_r}\sum_n \Big[2\partial_u q_r[\partial_n q_r \partial_n(q_r - \sum_t q_t) - e^{-2q_n}(\partial_n^2 q_r + \partial_n q_r \partial_n(q_r - 2q_n))] + \\
&+\partial_u[\partial_n q_r \partial_n(q_r - \sum_t q_t)] + e^{-2q_n}[2\partial_u q_n(\partial_n^2 q_r + \partial_n q_r \partial_n(q_r - 2q_n)) - \\
&-\partial_u(\partial_n^2 q_r + \partial_n q_r \partial_n(q_r - 2q_n))] - \\
&-2\partial_u q_r[\partial_n q_r \partial_n(q_r - \sum_t q_t) - e^{-2q_n}(\partial_n^2 q_r + \partial_n q_r \partial_n(q_r - 2q_n))]\Big] + \\
&+\delta_{ru}\, e^{2q_u}\Big[\sum_v e^{-2q_v}\partial_v q_u\Big(\partial_v \partial_s(q_v + q_s - \sum_t q_t) - \\
&-\frac{1}{2}[\partial_v q_s \partial_s(2q_v - \sum_t q_t) + \partial_s q_v \partial_v(2q_s - \sum_t q_t)] - \\
&-\frac{1}{2}\sum_n[\partial_v q_n \partial_s(q_n - q_v) + \partial_s q_n \partial_v(q_n - q_s)]\Big) + \\
&+\sum_n(\partial_s q_u[\partial_n q_s \partial_n(q_s - \sum_t q_t) - e^{-2q_n}(\partial_n^2 q_s + \partial_n q_s \partial_n(q_s - 2q_n))] - \\
&-\partial_s q_u[\partial_n q_r \partial_n(q_r - \sum_t q_t) - e^{-2q_n}(\partial_n^2 q_r + \partial_n q_r \partial_n(q_r - 2q_n))])\Big] + \\
&+\delta_{su}\, e^{2q_u}\Big[\sum_v e^{-2q_v}\partial_v q_u\Big(\partial_v \partial_r(q_v + q_r - \sum_t q_t) - \\
&-\frac{1}{2}[\partial_v q_r \partial_r(2q_v - \sum_t q_t) + \partial_r q_v \partial_v(2q_r - \sum_t q_t)] - \\
&-\frac{1}{2}\sum_n[\partial_v q_n \partial_r(q_n - q_v) + \partial_r q_n \partial_v(q_n - q_r)]\Big) + \\
&+\sum_n(\partial_r q_u[\partial_n q_r \partial_n(q_r - \sum_t q_t) - e^{-2q_n}(\partial_n^2 q_r + \partial_n q_r \partial_n(q_r - 2q_n)] - \\
&-\partial_r q_u[\partial_n q_s \partial_n(q_s - \sum_t q_t) - e^{-2q_n}(\partial_n^2 q_s + \partial_n q_s \partial_n(q_s - 2q_n))])\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\to_{r_{\bar a}\to 0} \frac{1}{2}\sum_{rsu}(\epsilon_{mur}\delta_{ns}+\epsilon_{nur}\delta_{ms})\cdot\\
&-\partial_u\partial_r\partial_s q+2(\partial_u q\partial_r\partial_s q+\partial_r q\partial_s\partial_u q+\partial_s q\partial_u\partial_r q)-4\partial_u q\partial_r q\partial_s q+\\
&+\delta_{rs}\sum_n\Big(\partial_u[\partial_n^2 q-(\partial_n q)^2]+2\partial_u q[\partial_n^2 q-(\partial_n q)^2]-2e^{2q}\partial_u(\partial_n q)^2\Big)-\\
&-\delta_{ru}\sum_v \partial_v q(\partial_v\partial_s q-\partial_v q\partial_s q)-\delta_{su}\sum_v \partial_v q(\partial_v\partial_r q-\partial_v q\partial_r q)=0,
\end{aligned}
$$

$$
\begin{aligned}
&\to_{q\to 0} \frac{1}{2}e^{2q}\sum_{rsu} e^{-2q_s}(\epsilon_{mur}\delta_{ns}+\epsilon_{nur}\delta_{ms})\cdot\\
&\Big(\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a r}+\gamma_{\bar a s})\partial_u\partial_r\partial_s r_{\bar a}-\frac{1}{3}\sum_{\bar a\bar b}(\gamma_{\bar a r}+\gamma_{\bar a s})(\gamma_{\bar b r}+\gamma_{\bar b s})\partial_u r_{\bar a}\partial_r\partial_s r_{\bar b}-\\
&-\frac{1}{3}\sum_{\bar a\bar b}\gamma_{\bar a u}[(\gamma_{\bar b u}+\gamma_{\bar b s})\partial_r r_{\bar a}\partial_u\partial_s r_{\bar b}+(\gamma_{\bar b u}+\gamma_{\bar b r})\partial_s r_{\bar a}\partial_u\partial_r r_{\bar b}]-\\
&-\frac{2}{3}\sum_{\bar a\bar b}\gamma_{\bar a s}\gamma_{\bar b r}\partial_u[\partial_r r_{\bar a}\partial_s r_{\bar b}]-\\
&-\frac{1}{6}\sum_{\bar a\bar b}\sum_n \gamma_{\bar a n}\partial_u[(\gamma_{\bar b n}-\gamma_{\bar b r})\partial_r r_{\bar a}\partial_s r_{\bar b}+(\gamma_{\bar b n}-\gamma_{\bar b s})\partial_s r_{\bar a}\partial_r r_{\bar b}]+\\
&+\frac{2}{3\sqrt{3}}\sum_{\bar a\bar b\bar c}(\gamma_{\bar a r}+\gamma_{\bar s})(\gamma_{\bar b u}\gamma_{\bar c u}+\gamma_{\bar b s}\gamma_{\bar c r})\partial_u r_{\bar a}\partial_r r_{\bar b}\partial_s r_{\bar c}+\\
&+\frac{1}{6\sqrt{3}}\sum_{\bar a\bar b\bar c}\sum_n \gamma_{\bar b n}\Big[(\gamma_{\bar a r}+\gamma_{\bar a s})\partial_u r_{\bar a}[(\gamma_{\bar c n}-\gamma_{\bar c r})\partial_r r_{\bar b}\partial_s r_{\bar c}+(\gamma_{\bar c n}-\gamma_{\bar c s})\partial_s r_{\bar b}\partial_r r_{\bar c}]+\\
&+\gamma_{\bar a u}(\,\partial_r r_{\bar a}[(\gamma_{\bar c n}-\gamma_{\bar c u})\partial_u r_{\bar b}\partial_s r_{\bar c}+(\gamma_{\bar c n}-\gamma_{\bar c s})\partial_s r_{\bar b}\partial_u r_{\bar c}]+\\
&+\partial_s r_{\bar a}[(\gamma_{\bar c n}-\gamma_{\bar c u})\partial_u r_{\bar b}\partial_r r_{\bar c}+(\gamma_{\bar c n}-\gamma_{\bar c r})\partial_r r_{\bar b}\partial_u r_{\bar c}]\,)\Big]+\\
&+\frac{1}{3\sqrt{3}}\delta_{rs}e^{\frac{2}{\sqrt{3}}\sum_{\bar e}\gamma_{\bar e r}r_{\bar e}}\sum_n\sum_{\bar a}\Big(2\gamma_{\bar a r}\partial_u r_{\bar a}\sum_{\bar b\bar c}\gamma_{\bar b r}\gamma_{\bar c r}\partial_n r_{\bar b}\partial_n r_{\bar c}-\\
&-e^{-\frac{2}{\sqrt{3}}\sum_{\bar d}\gamma_{\bar d n}r_{\bar d}}\gamma_{\bar a r}[\sqrt{3}\partial_n^2 r_{\bar a}+(\gamma_{\bar b r}-2\gamma_{\bar b n}\partial_n r_{\bar a}\partial_n r_{\bar b})]+\\
&+\sqrt{3}\sum_{\bar b}\gamma_{\bar a r}\gamma_{\bar b r}\partial_u[\partial_n r_{\bar a}\partial_n r_{\bar b}]+\\
&+e^{-\frac{2}{\sqrt{3}}\sum_{\bar d}\gamma_{\bar d n}r_{\bar d}}[2\gamma_{\bar a n}\partial_u r_{\bar a}\sum_{\bar b}\gamma_{\bar b r}(\sqrt{3}\partial_n^2 r_{\bar B}+\\
&+\sum_{\bar c}(\gamma_{\bar c r}-2\gamma_{\bar c n})\partial_n r_{\bar b}\partial_n r_{\bar c})-\gamma_{\bar a r}\partial_u(\sqrt{3}\partial_n^2 r_{\bar a}+\sum_{\bar b}(\gamma_{\bar b r}-2\gamma_{\bar b n})\partial_n r_{\bar a}\partial_n r_{\bar b})]-\\
&-2\gamma_{\bar a r}\partial_u r_{\bar a}\sum_{\bar b}[\sum_{\bar c}\gamma_{\bar b r}\gamma_{\bar c r}\partial_n r_{\bar b}\partial_n r_{\bar c}-\\
&-e^{-\frac{2}{\sqrt{3}}\sum_{\bar d}\gamma_{\bar d n}r_{\bar d}}\gamma_{\bar b r}(\sqrt{3}\partial_n^2 r_{\bar b}+\sum_{\bar c}(\gamma_{\bar c r}-2\gamma_{\bar c n})\partial_n r_{\bar b}\partial_n r_{\bar c})]\Big)+\\
&+\frac{1}{3}\delta_{ru}e^{\frac{2}{\sqrt{3}}\sum_{\bar e}\gamma_{\bar e u}r_{\bar e}}\sum_{\bar a}\Big(\sum_v e^{-\frac{2}{\sqrt{3}}\sum_{\bar d}\gamma_{\bar d v}r_{\bar d}}\sum_{\bar b}\gamma_{\bar a u}\partial_v r_{\bar a}\cdot
\end{aligned}
$$

$$
\begin{aligned}
&\Big[(\gamma_{\bar{b}v} + \gamma_{\bar{b}s})\partial_v\partial_s r_{\bar{b}} - \frac{2}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{b}s}\gamma_{\bar{c}v}\partial_v r_{\bar{b}}\partial_s r_{\bar{c}} - \\
&-\frac{1}{2\sqrt{3}}\sum_n \gamma_{\bar{b}n}\sum_{\bar{c}}[(\gamma_{\bar{c}n} - \gamma_{\bar{c}v})\partial_v r_{\bar{b}}\partial_s r_{\bar{c}} + (\gamma_{\bar{c}n} - \gamma_{\bar{c}s})\partial_s r_{\bar{b}}\partial_v r_{\bar{c}}]\Big] + \\
&+\frac{1}{\sqrt{3}}\sum_n(\gamma_{\bar{a}u}\partial_s r_{\bar{a}}[\sum_{\bar{b}\bar{c}}\gamma_{\bar{b}s}\gamma_{\bar{c}s}\partial_n r_{\bar{b}}\partial_n r_{\bar{c}} - e^{-\frac{2}{\sqrt{3}}\sum_{\bar{d}}\gamma_{\bar{d}n}r_{\bar{d}}}\sum_{\bar{b}}\gamma_{\bar{b}s}(\sqrt{3}\partial_n^2 r_{\bar{b}} + \\
&+\sum_{\bar{c}}(\gamma_{\bar{c}s} - 2\gamma_{\bar{c}n})\partial_n r_{\bar{b}}\partial_n r_{\bar{c}})] - \gamma_{\bar{a}u}\partial_s r_{\bar{a}}\sum_{\bar{b}}[\sum_{\bar{c}}\gamma_{\bar{b}r}\gamma_{\bar{c}r}\partial_n r_{\bar{b}}\partial_n r_{\bar{c}} - \\
&-e^{-\frac{2}{\sqrt{3}}\sum_{\bar{d}}\gamma_{\bar{d}n}r_{\bar{d}}}\gamma_{\bar{b}r}(\sqrt{3}\partial_n^2 r_{\bar{b}} + \sum_{\bar{c}}(\gamma_{\bar{c}r} - 2\gamma_{\bar{c}n})\partial_n r_{\bar{b}}\partial_n r_{\bar{c}})])\} + \\
&+\frac{1}{3}\delta_{su}e^{\frac{2}{\sqrt{3}}\sum_{\bar{e}}\gamma_{\bar{e}u}r_{\bar{e}}}\sum_{\bar{a}}\Big(\sum_v e^{-\frac{2}{\sqrt{3}}\sum_{\bar{d}}\gamma_{\bar{d}v}r_{\bar{d}}}\sum_{\bar{b}}\gamma_{\bar{a}u}\partial_v r_{\bar{a}}\cdot \\
&\Big[(\gamma_{\bar{b}v} + \gamma_{\bar{b}r})\partial_v\partial_r r_{\bar{b}} - \frac{2}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{b}r}\gamma_{\bar{c}v}\partial_v r_{\bar{b}}\partial_s r_{\bar{c}} - \\
&-\frac{1}{2\sqrt{3}}\sum_n \gamma_{\bar{b}n}\sum_{\bar{c}}[(\gamma_{\bar{c}n} - \gamma_{\bar{c}v})\partial_v r_{\bar{b}}\partial_r r_{\bar{c}} + (\gamma_{\bar{c}n} - \gamma_{\bar{c}r})\partial_r r_{\bar{b}}\partial_v r_{\bar{c}}]\Big] + \\
&+\frac{1}{\sqrt{3}}\sum_n(\gamma_{\bar{a}u}\partial_r r_{\bar{a}}[\sum_{\bar{b}\bar{c}}\gamma_{\bar{b}r}\gamma_{\bar{c}r}\partial_n r_{\bar{b}}\partial_n r_{\bar{c}} - e^{-\frac{2}{\sqrt{3}}\sum_{\bar{d}}\gamma_{\bar{d}n}r_{\bar{d}}}\sum_{\bar{b}}\gamma_{\bar{b}r}(\sqrt{3}\partial_n^2 r_{\bar{b}} + \\
&+\sum_{\bar{c}}(\gamma_{\bar{c}r} - 2\gamma_{\bar{c}n})\partial_n r_{\bar{b}}\partial_n r_{\bar{c}})] - \gamma_{\bar{a}u}\partial_r r_{\bar{a}}\sum_{\bar{b}}[\sum_{\bar{c}}\gamma_{\bar{b}s}\gamma_{\bar{c}s}\partial_n r_{\bar{b}}\partial_n r_{\bar{c}} - \\
&-e^{-\frac{2}{\sqrt{3}}\sum_{\bar{d}}\gamma_{\bar{d}n}r_{\bar{d}}}\gamma_{\bar{b}s}(\sqrt{3}\partial_n^2 r_{\bar{b}} + \sum_{\bar{c}}(\gamma_{\bar{c}s} - 2\gamma_{\bar{c}n})\partial_n r_{\bar{b}}\partial_n r_{\bar{c}})])\Big)\Big).
\end{aligned}
\tag{D2}
$$

Since the condition $r_{\bar{a}} = 0$ corresponds to conformally flat 3-manifolds Σ_τ, the Cotton-York conformal tensor vanishes in the limit $r_a \to 0$.

APPENDIX E: 4-TENSORS IN THE FINAL CANONICAL BASIS.

From the results of Appendix B of I and Eq.(99) for ${}^3\hat{\tilde{\pi}}^r_{(a)}$, we get the following expressions for the reconstruction of 4-tensors on M^4 [here $N = N_{(as)} + n$ and $N_{(a)} = {}^3\hat{e}^r_{(a)} N_r = {}^3\hat{e}^r_{(a)} [N_{(as)r} + n_r] = e^{-q} \sum_r \delta^r_{(a)} e^{-\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} [N_{(as)r} + n_r]$ [$n_r = 0$ as after Eq.(93) will no more hold after the addition of surface terms to the Dirac Hamiltonian to make it differentiable, see Ref. [6]; when $n_r \neq 0$, one has to replace $N_{(as)r}$ with $N_{(as)r} + n_r$ in the following formulas] are the total lapse and shift functions; only in the Christoffel symbols we shall put the explicit expression of $N_{(a)}$]:

$$
\begin{aligned}
{}^4\hat{\Gamma}^\tau_{\tau\tau} &= \frac{1}{N}\Big[\partial_\tau N + e^{-2q} \sum_r e^{-\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} N_{(as)r} \partial_r N - \\
&- \frac{\epsilon}{4k} e^{-4q}\, {}^3G_{o(a)(b)(c)(d)} \sum_{mn} e^{-\frac{1}{\sqrt{3}} \sum_{\bar{a}} (\gamma_{\bar{a}m} + \gamma_{\bar{a}n}) r_{\bar{a}}} \\
&\quad \delta_{(a)m} \delta_{(b)n} N_{(as)m} N_{(as)n} \sum_u \delta_{(c)u} e^{\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}u} r_{\bar{a}}}\, {}^3\hat{\tilde{\pi}}^u_{(d)}\Big] = \\
&\quad =_{N_{(a)}=0} \qquad \frac{1}{N} \partial_\tau N, \\
{}^4\hat{\Gamma}^\tau_{r\tau} &= {}^4\hat{\Gamma}^\tau_{\tau r} = \frac{1}{N}\Big[\partial_r N - \frac{\epsilon}{4k}\, {}^3G_{o(a)(b)(c)(d)} \delta_{(a)r} \\
&\quad \sum_{su} e^{\frac{1}{\sqrt{3}} \sum_{\bar{a}} (\gamma_{\bar{a}r} - \gamma_{\bar{a}s} + \gamma_{\bar{a}u}) r_{\bar{a}}} \delta_{(b)s} N_{(as)s} \delta_{(c)u}\, {}^3\hat{\tilde{\pi}}^u_{(d)}\Big] = \\
&\quad =_{N_{(a)}=0} \qquad \frac{1}{N} \partial_r N, \\
{}^4\hat{\Gamma}^\tau_{rs} &= {}^4\hat{\Gamma}^\tau_{sr} = -\frac{\epsilon}{4kN} \sum_u e^{\frac{1}{\sqrt{3}} \sum_{\bar{a}} (\gamma_{\bar{a}r} + \gamma_{\bar{a}s} + \gamma_{\bar{a}u}) r_{\bar{a}}}\, {}^3G_{o(a)(b)(c)(d)} \delta_{(a)r} \delta_{(b)s} \delta_{(c)u}\, {}^3\hat{\tilde{\pi}}^u_{(d)}, \\
{}^4\hat{\Gamma}^u_{\tau\tau} &= \Big[\partial_\tau\Big(e^{-q-\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}u} r_{\bar{a}}} N_{(as)u}\Big) - \\
&- \frac{\partial_\tau N}{N} e^{-q-\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}u} r_{\bar{a}}} N_{(as)u}\Big] e^{-q-\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}u} r_{\bar{a}}} + \\
&+ N e^{-2q}\Big(\delta_{(a)(b)} - e^{-2q} \sum_{rs} e^{-\frac{1}{\sqrt{3}} \sum_{\bar{a}} (\gamma_{\bar{a}r} + \gamma_{\bar{a}s}) r_{\bar{a}}} \delta_{(a)r} \delta_{(b)s} \frac{N_{(as)r} N_{(as)s}}{N^2}\Big) \\
&\quad \sum_v e^{-\frac{1}{\sqrt{3}} \sum_{\bar{a}} (\gamma_{\bar{a}u} + \gamma_{\bar{a}v}) r_{\bar{a}}} \delta^u_{(a)} \delta^v_{(b)} \partial_v N + \\
&+ e^{-2q} \sum_v e^{-\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}v} r_{\bar{a}}} N_{(as)v} (e^{-2q-\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}u} r_{\bar{a}}} N_{(as)u})_{|v} - \\
&- \frac{\epsilon N}{2k} e^{-4q}\, {}^3G_{o(a)(b)(c)(d)} \sum_w e^{-\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}w} r_{\bar{a}}} \delta_{(a)w} N_{(as)w} (\delta_{(a)(b)} - \\
&- e^{-2q} \sum_{rs} e^{-\frac{1}{\sqrt{3}} \sum_{\bar{a}} (\gamma_{\bar{a}r} + \gamma_{\bar{a}s}) r_{\bar{a}}} \frac{N_{(as)r} N_{(as)s}}{2N^2}) \sum_v \delta^u_{(a)} e^{-\frac{1}{\sqrt{3}} \sum_{\bar{a}} (\gamma_{\bar{a}u} - \gamma_{\bar{a}v}) r_{\bar{a}}} \delta_{(c)v}\, {}^3\hat{\tilde{\pi}}^v_{(d)} - \\
&- e^{-2q} \sum_r e^{-\frac{1}{\sqrt{3}} \sum_{\bar{a}} (\gamma_{\bar{a}r} + \gamma_{\bar{a}u}) r_{\bar{a}}} N_{(as)r} \\
&\quad \Big[\frac{\epsilon N}{4k} e^{-2q} \delta_{(a)r} \delta^u_{(b)}\, {}^3G_{o(a)(b)(c)(d)} \sum_s e^{\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}s} r_{\bar{a}}} \delta_{(c)s}\, {}^3\hat{\tilde{\pi}}^s_{(d)} +
\end{aligned}
$$

$$
\begin{aligned}
&+ e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}\Big(e^{-2q}\sum_w e^{-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar aw}r_{\bar a}}N_{(as)w}\partial_w[e^{q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}]+\\
&+ e^{q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar au}r_{\bar a}}\partial_r[e^{-2q-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar au}r_{\bar a}}N_{(as)u}]\Big)\Big] =\\
&=_{N_{(a)}=0} \qquad N e^{-2(q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar au}r_{\bar a})}\partial_u N,
\end{aligned}
$$

$$
\begin{aligned}
{}^4\hat{\Gamma}^u_{r\tau} &= {}^4\hat{\Gamma}^u_{\tau r} = e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar au}r_{\bar a}}\Big[(e^{-q}\sum_s e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar as}r_{\bar a}}\delta_{(a)s}N_{(as)s})_{|r} -\\
&- e^{-q}\sum_s e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar as}r_{\bar a}}\delta_{(a)s}N_{(as)s}\frac{\partial_r N}{N}\Big] -\\
&- \frac{\epsilon N}{4k}e^{-2q}(\delta_{(a)(b)} - e^{-2q}\sum_{rs}e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar ar}+\gamma_{\bar as})r_{\bar a}}\delta_{(a)r}\delta_{(b)s}\frac{N_{(as)r}N_{(as)s}}{N^2})\\
&\quad e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar au}r_{\bar a}}\delta^u_{(a)}\,{}^3G_{o(b)(c)(d)(e)}\sum_w e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar ar}+\gamma_{\bar aw})r_{\bar a}}\delta_{(c)r}\delta_{(d)w}\,{}^3\hat{\tilde{\pi}}^w_{(e)} =\\
&\quad =_{N_{(a)}=0} \qquad -\frac{\epsilon N}{4k}\sum_{uv}e^{-2q+\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar ar}+\gamma_{\bar av}+\gamma_{\bar au})r_{\bar a}}\,{}^3G_{o(a)(b)(c)(d)}\delta^u_{(a)}\delta_{(b)r}\delta_{(c)v}\,{}^3\hat{\tilde{\pi}}^v_{(d)},\\
{}^4\hat{\Gamma}^u_{rs} &= {}^3\hat{\Gamma}^u_{rs} + \frac{\epsilon}{4k}e^{-2q-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar au}r_{\bar a}}N_{(as)u}\,{}^3G_{o(a)(b)(c)(d)}\\
&\quad \sum_v e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar ar}+\gamma_{\bar as}+\gamma_{\bar av})r_{\bar a}}\delta_{(a)r}\delta_{(b)s}\delta_{(c)v}\,{}^3\hat{\tilde{\pi}}^v_{(d)} =\\
&\quad =_{N_{(a)}=0} \qquad {}^3\hat{\Gamma}^u_{rs},
\end{aligned}
$$

$$
\begin{aligned}
{}^4\hat{\overset{\circ}{\omega}}_{\tau(o)(a)} &= -{}^4\hat{\overset{\circ}{\omega}}_{\tau(a)(o)} = -\epsilon\sum_r \delta_{(a)r}e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a})}\partial_r N -\\
&- \frac{e^{-2q}}{4k}{}^3G_{o(a)(b)(c)(d)}N_{(b)}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar au}r_{\bar a}}\delta_{(c)u}\,{}^3\hat{\tilde{\pi}}^u_{(d)} =\\
&\quad =_{N_{(a)}=0} \qquad -\epsilon\sum_r \delta_{(a)r}e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a})}\partial_r N,\\
{}^4\hat{\overset{\circ}{\omega}}_{\tau(a)(b)} &= -{}^4\hat{\overset{\circ}{\omega}}_{\tau(b)(a)} \overset{\circ}{=} -\epsilon\sum_r e^{-q}\,{}^3\hat{\omega}_{r(a)(b)}e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}\delta^r_{(c)}N_{(c)} =\\
&\quad =_{N_{(a)}=0} \qquad 0,\\
{}^4\hat{\overset{\circ}{\omega}}_{r(o)(a)} &= -{}^4\hat{\overset{\circ}{\omega}}_{r(a)(o)} = -\frac{1}{4k}\sum_u e^{-q+\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar ar}+\gamma_{\bar au})r_{\bar a}}\,{}^3G_{o(a)(b)(c)(d)}\delta_{(b)r}\delta_{(c)u}\,{}^3\hat{\tilde{\pi}}^u_{(d)},\\
{}^4\hat{\overset{\circ}{\omega}}_{r(a)(b)} &= -{}^4\hat{\overset{\circ}{\omega}}_{r(b)(a)} = -\epsilon\,{}^3\hat{\omega}_{r(a)(b)},
\end{aligned}
$$

$$
\begin{aligned}
{}^4\hat{\overset{\circ}{\Omega}}_{rs(a)(b)} &= -\epsilon\Big[{}^3\hat{\Omega}_{rs(a)(b)} + \frac{e^{-2q}}{4k}\,{}^3G_{o(a)(c)(d)(e)}\,{}^3G_{o(b)(f)(g)(h)}\cdot\\
&\quad \sum_{uv}e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar ar}+\gamma_{\bar as}+\gamma_{\bar au}+\gamma_{\bar av})r_{\bar a}}(\delta_{(c)r}\delta_{(f)s}-\delta_{(c)s}\delta_{(f)r})\delta_{(d)u}\,{}^3\hat{\tilde{\pi}}^u_{(e)}\,\delta_{(g)v}\,{}^3\hat{\tilde{\pi}}^v_{(h)}\Big],\\
{}^4\hat{\overset{\circ}{\Omega}}_{rs(o)(a)} &= \frac{1}{N}\sum_v e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar av}r_{\bar a})}\delta_{(a)v}\Big({}^4\hat{R}_{\tau vrs} - N_{(b)}\sum_u e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar bu}r_{\bar b}}\delta_{(b)u}\,{}^4\hat{R}_{uvrs}\Big) =\\
&= \frac{1}{4k}\sum_u e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar au}r_{\bar a})}\delta_{(a)u}
\end{aligned}
$$

$$
\begin{aligned}
&\Big[\Big(e^{\frac{1}{\sqrt{3}}\sum_{\bar{b}}(\gamma_{\bar{b}r}+\gamma_{\bar{b}u}+\gamma_{\bar{b}v})r_{\bar{b}}}\,{}^3G_{o(b)(c)(d)(e)}\ \delta_{(b)r}\delta_{(c)u}\delta_{(d)v}\ {}^3\hat{\tilde{\pi}}^v_{(e)}\Big)_{|s} -\\
&-\Big(e^{\frac{1}{\sqrt{3}}\sum_{\bar{b}}(\gamma_{\bar{b}s}+\gamma_{\bar{b}u}+\gamma_{\bar{b}v})r_{\bar{b}}}\,{}^3G_{o(b)(c)(d)(e)}\ \delta_{(b)s}\delta_{(c)u}\delta_{(d)v}\ {}^3\hat{\tilde{\pi}}^v_{(e)}\Big)_{|r}\Big],\\
{}^4\hat{\mathring{\Omega}}_{\tau r(a)(b)} &= \sum_{uv} e^{-(2q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}u}+\gamma_{\bar{a}v})r_{\bar{a}})}\delta_{(a)u}\delta_{(b)v)}\ {}^4\hat{R}_{uv\tau r}\,\mathring{=}\\
&\mathring{=} -\epsilon\Big(\partial_\tau\ {}^3\hat{\omega}_{r(a)(b)} + \frac{1}{2}\Big[\epsilon_{(a)(b)(c)}\epsilon_{(d)(e)(f)} - \epsilon_{(a)(b)(d)}\epsilon_{(c)(e)(f)}\Big]\cdot\\
&\sum_s e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}})}\delta_{(c)s}\\
&\Big[\frac{\epsilon N}{4k}\sum_v e^{-q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}(\gamma_{\bar{b}s}+\gamma_{\bar{b}v})r_{\bar{b}}}\,{}^3G_{o(d)(l)(m)(n)}\ \delta_{(l)s}\delta_{(m)v}\ {}^3\hat{\tilde{\pi}}^v_{(n)} +\\
&+ N_{(l)}\sum_u e^{(q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}r_{\bar{b}})}\delta_{(l)u}\partial_u\Big(\delta_{(d)s}e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}s}r_{\bar{c}}}\Big) +\\
&+ \sum_u e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}r_{\bar{b}}}\delta_{(d)u}\partial_s\Big(N_{(l)}\delta_{(l)u}e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u}r_{\bar{c}})}\Big) +\\
&+ \epsilon_{(d)(m)(n)}\hat{\mu}_{(m)}\delta_{(n)s}e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}s}r_{\bar{b}}} -\\
&- N_{(g)}\sum_u e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}r_{\bar{b}})}\delta_{(g)u}\partial_u\Big(\delta_{(d)s}e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}s}r_{\bar{c}}}\Big) -\\
&- \sum_u e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}r_{\bar{b}}}\delta_{(d)u}\partial_s\Big(N_{(g)}\delta_{(g)u}e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u}r_{\bar{c}})}\Big)\Big]{}^3\hat{\omega}_{r(e)(f)} +\\
&+ N_{(c)}\delta_{(c)s}e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}s}r_{\bar{b}})}\ \big[{}^3\hat{\omega}_s, {}^3\hat{\omega}_r\big]_{(a)(b)} +\\
&+ \frac{\epsilon}{4k}\sum_u e^{-2q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}r}r_{\bar{b}}}\,{}^3G_{o(c)(d)(e)(f)}\delta_{(c)r}\delta_{(e)u}\ {}^3\hat{\tilde{\pi}}^u_{(f)}\\
&\Big(\delta_{(a)(d)}\delta_{(b)u} - \delta_{(b)(d)}\delta_{(a)u}\Big)\partial_u N +\\
&+ \frac{1}{(4k)^2}\Big(\delta_{(a)(l)}\delta_{(b)(d)} - \delta_{(a)(d)}\delta_{(b)(l)}\Big){}^3G_{o(d)(e)(f)(g)}\ {}^3G_{o(h)(l)(m)(n)}\ \cdot\\
&\cdot\ \sum_{wv} e^{-3q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}(\gamma_{\bar{b}r}+\gamma_{\bar{b}w}+\gamma_{\bar{b}v})r_{\bar{b}}}\delta_{(h)r}N_{(e)}\delta_{(f)w}\ {}^3\hat{\tilde{\pi}}^w_{(g)}\delta_{(m)v}\ {}^3\hat{\tilde{\pi}}^v_{(n)}\Big),\\
{}^4\hat{\mathring{\Omega}}_{\tau r(o)(a)} &\mathring{=} \frac{1}{N}\sum_u e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}})}\delta_{(a)u}\Big[{}^4\hat{R}_{\tau u\tau r} - N_{(b)}\sum_s e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}s}r_{\bar{b}})}\delta_{(b)s}\ {}^4\hat{R}_{su\tau r}\Big]\,\mathring{=}\\
&\mathring{=} -\epsilon\sum_s e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}})}\delta_{(a)s}\Big[\partial_\tau\ {}^3\hat{K}_{rs} + N_{|s|r} -\\
&- \frac{\epsilon}{4k}\sum_{uw} e^{\frac{1}{\sqrt{3}}\sum_{\bar{b}}(\gamma_{\bar{b}u}+\gamma_{\bar{b}w})r_{\bar{b}}}\,{}^3G_{o(c)(d)(e)(f)}\delta_{(d)u}\delta_{(e)w}\ {}^3\hat{\tilde{\pi}}^w_{(f)}\\
&\Big(e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}r}r_{\bar{c}}}\delta_{(c)r}\Big(N_{(b)}e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{d}}\gamma_{\bar{d}u}r_{\bar{d}})}\delta_{(b)u}\Big)_{|s} +\\
&+ e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}s}r_{\bar{c}}}\delta_{(c)s}\Big(N_{(b)}e^{-(q+\frac{1}{\sqrt{3}}\sum_{\bar{d}}\gamma_{\bar{d}u}r_{\bar{d}}}\delta_{(b)u}\Big)_{|r}\Big) -\\
&- \frac{\epsilon}{4k}\sum_{usw} N_{(b)}e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}r_{\bar{b}}}\delta_{(b)u}\\
&\Big(e^{3q+\frac{1}{\sqrt{3}}\sum_{\bar{c}}(\gamma_{\bar{c}s}+\gamma_{\bar{c}u}+\gamma_{\bar{c}w})r_{\bar{c}}}\,{}^3G_{o(c)(d)(e)(f)}\ \delta_{(c)s}\delta_{(d)u}\delta_{(e)w}\ {}^3\hat{\tilde{\pi}}^w_{(f)}\Big)_{|r}\Big],
\end{aligned}
$$

$$
\begin{aligned}
{}^4\hat{R}_{rsuv} &= \delta_{(a)r}\delta_{(b)s}e^{2q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}+\gamma_{\bar{a}s})r_{\bar{a}}}\,{}^4\hat{\overset{\circ}{\Omega}}_{uv(a)(b)} = \\
&= -{}^3\hat{R}_{rsuv} + \frac{N^2}{16k^2}\sum_{tw} e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}+\gamma_{\bar{a}s}+\gamma_{\bar{a}u}+\gamma_{\bar{a}v}+\gamma_{\bar{a}t}+\gamma_{\bar{a}w})r_{\bar{a}}}\,.\\
&\quad {}^3G_{o(a)(b)(c)(d)}\,{}^3G_{o(e)(f)(g)(h)}\\
&\quad \cdot\delta_{(a)r}\delta_{(e)s}\Big(\delta_{(b)u}\delta_{(f)v}-\delta_{(b)v}\delta_{(f)u}\Big)\delta_{(c)t}\delta_{(g)w}\,{}^3\hat{\tilde{\pi}}^t_{(d)}\,{}^3\hat{\tilde{\pi}}^w_{(h)},\\
{}^4\hat{R}_{\tau ruv} &= N\delta_{(a)r}e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\,{}^4\hat{\overset{\circ}{\Omega}}_{uv(o)(a)},\\
{}^4\hat{R}_{\tau r\tau s} &= N\delta_{(a)r}e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\,{}^4\hat{\overset{\circ}{\Omega}}_{\tau s(o)(a)},\\
\\
{}^4\hat{R}_{\tau\tau} &= -\epsilon\sum_r e^{-2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}})}\,{}^4\hat{R}_{r\tau r\tau},\\
{}^4\hat{R}_{\tau r} &= {}^4\hat{R}_{r\tau} = -\epsilon\sum_u e^{-2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}})}\,{}^4\hat{R}_{u\tau ur},\\
{}^4\hat{R}_{rs} &= {}^4\hat{R}_{sr} = \frac{\epsilon}{N^2}{}^4\hat{R}_{\tau r\tau s} - \epsilon\sum_u e^{-2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}})}\,{}^4\hat{R}_{urus},\\
{}^4\hat{R} &= \frac{\epsilon}{N^2}{}^4\hat{R}_{\tau\tau} - \epsilon\sum_r e^{-2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}})}\,{}^4\hat{R}_{rr},\\
\\
{}^4\hat{C}_{rsuv} &= {}^4\hat{R}_{rsuv} + \frac{\epsilon}{2}\Big[e^{2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}})}(\delta_{rv}\,{}^4\hat{R}_{su}-\delta_{ru}\,{}^4\hat{R}_{sv}) + \\
&\quad + e^{2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}})}(\delta_{su}\,{}^4\hat{R}_{rv}-\delta_{sv}\,{}^4\hat{R}_{ru})\Big] + \\
&\quad + \frac{1}{6}e^{2(2q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}+\gamma_{\bar{a}s})r_{\bar{a}})}(\delta_{ru}\delta_{sv}-\delta_{rv}\delta_{su})\,{}^4\hat{R},\\
{}^4\hat{C}_{\tau ruv} &= {}^4\hat{R}_{\tau ruv} + \frac{\epsilon}{2}e^{2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}})}(\delta_{ru}\,{}^4\hat{R}_{\tau v}-\delta_{rv}\,{}^4\hat{R}_{\tau u}),\\
{}^4\hat{C}_{\tau r\tau s} &= {}^4\hat{R}_{\tau r\tau s} + \frac{1}{2}\Big[N^2\,{}^4\hat{R}_{rs} - \epsilon e^{2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}})}\delta_{rs}\,{}^4\hat{R}_{\tau\tau}\Big] - \\
&\quad - \frac{1}{6}N^2 e^{2(q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}})}\delta_{rs}\,{}^4\hat{R},
\end{aligned}
\tag{E1}
$$

To get $\partial_\tau\,{}^3\hat{\omega}_{r(a)(b)}$ and $\partial_\tau\,{}^3\hat{K}_{rs}$ we need Eqs.(65) of I, where in Appendix B it is noted that $\partial_\tau\,{}^3\hat{K}_{rs}$ needs the use of the second half of Hamilton equations.

The York almost canonical basis of Appendix C takes the form [$\mathcal{T}(\tau,\vec{\sigma})$ is the "extrinsic internal time" proportional to the "mean extrinsic curvature"; ${}^3\sigma_{rs}(\tau,\vec{\sigma})$ is the "conformal metric" [$det\,({}^3\sigma_{rs})=1$], which is a density of weight=-2/3 like the momentum ${}^3\tilde{\Pi}^{rs}_A(\tau,\vec{\sigma})$]

$$
\begin{aligned}
\mathcal{T}(\tau,\vec{\sigma}) &= -\frac{4}{3}\epsilon k\,{}^3K(\tau,\vec{\sigma}) = [\frac{2\,{}^3g_{rs}\,{}^3\tilde{\Pi}^{rs}}{3\sqrt{\gamma}}](\tau,\vec{\sigma}) \mapsto \hat{\mathcal{T}}(\tau,\vec{\sigma}) = \\
&\quad = \frac{\epsilon}{3}e^{q(\tau,\vec{\sigma})}\sum_r[e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}](\tau,\vec{\sigma})\int d^3\sigma_1\mathcal{K}^r_{(b)s}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}})\cdot\\
&\quad\cdot\ (e^{-q-\frac{1}{3}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}})(\tau,\vec{\sigma}_1)\Big[\frac{1}{3}\rho+\sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}s}\pi_{\bar{b}}\Big](\tau,\vec{\sigma}_1),\\
\mathcal{P}_{\mathcal{T}}(\tau,\vec{\sigma}) &= -\sqrt{\gamma} \mapsto \hat{\mathcal{P}}_{\mathcal{T}}(\tau,\vec{\sigma}) = -e^{3q(\tau,\vec{\sigma})},
\end{aligned}
$$

$$
\begin{aligned}
{}^3\sigma_{rs}(\tau,\vec{\sigma}) &= [\frac{{}^3g_{rs}}{\gamma^{1/3}}](\tau,\vec{\sigma}) \mapsto {}^3\hat{\sigma}_{rs}(\tau,\vec{\sigma}) = {}^3g^Y_{rs}(\tau,\vec{\sigma}) = e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}(\tau,\vec{\sigma})}\,\delta_{rs},\\
{}^3\tilde{\Pi}^{rs}_A(\tau,\vec{\sigma}) &= \Big[\gamma^{1/3}({}^3\tilde{\Pi}^{rs} - \frac{1}{3}\,{}^3g^{rs}\,{}^3\tilde{\Pi})\Big](\tau,\vec{\sigma}) \mapsto\\
&\mapsto \frac{\epsilon}{4}\Big(e^{4q}\Big[e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\delta^r_{(a)}\;{}^3\hat{\tilde{\pi}}^s_{(a)} +\\
&+e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\delta^s_{(a)}\;{}^3\hat{\tilde{\pi}}^r_{(a)} - \frac{2}{3}e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}u}-2\gamma_{\bar{a}r})r_{\bar{a}}}\delta^{rs}\delta^u_{(a)}\;{}^3\hat{\tilde{\pi}}^u_{(a)}\Big]\Big)(\tau,\vec{\sigma}) \qquad \text{(E2)}
\end{aligned}
$$

Using Eqs.(68) and (69) of I, Ashtekar's variables become

$$
\begin{aligned}
&{}^3\tilde{h}^r_{(a)}(\tau,\vec{\sigma}) \mapsto {}^3\hat{\tilde{h}}^r_{(a)}(\tau,\vec{\sigma}) = \delta^r_{(a)}\Big[e^{2q}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\Big](\tau,\vec{\sigma}),\\
&{}^3A_{(a)r}(\tau,\vec{\sigma}) \mapsto {}^3\hat{A}_{(a)r}(\tau,\vec{\sigma}) = \frac{1}{2k}(e^{2q+\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}})(\tau,\vec{\sigma})\\
&\int d^3\sigma_1\,\mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{c}}]\,\Big[e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{d}}\gamma_{\bar{d}s}r_{\bar{d}}}(\frac{\rho}{3}+\sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}s}\pi_{\bar{b}})\Big](\tau,\vec{\sigma}_1) +\\
&+ i\epsilon_{(a)(b)(c)}\delta_{(b)r}\delta_{(c)u}\Big[e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}-\gamma_{\bar{a}u})r_{\bar{a}}}(\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}r}\partial_u r_{\bar{b}})\Big](\tau,\vec{\sigma}). \qquad \text{(E3)}
\end{aligned}
$$

With the further gauge fixing $\tilde{\lambda}_r(\tau)=0$ [i.e. when $n_r=0$ and $N_{(as)(a)}=0$] the 4-geodesic and 4-geodesic deviation equations given at the end of Appendix A of I become respectively in the 3-orthogonal gauges

$$
\begin{aligned}
&\frac{d^2\tau(s)}{ds^2} + \frac{\partial_\tau N}{N} + 2\frac{\partial_r N}{N}\frac{d\tau(s)}{ds}\frac{d\sigma^r(s)}{ds} -\\
&\quad - \frac{\epsilon}{4kN}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}+\gamma_{\bar{a}s}+\gamma_{\bar{a}u})r_{\bar{a}}}\,{}^3G_{o(a)(b)(c)(d)}\delta_{(a)r}\delta_{(b)s}\delta_{(c)u}\;{}^3\hat{\tilde{\pi}}^u_{(d)}\frac{d\sigma^r(s)}{ds}\frac{d\sigma^s(s)}{ds} = 0,\\
&\frac{d^2\sigma^u(s)}{ds^2} + N\phi^{-4}e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}\partial_u N(\frac{d\tau(s)}{ds})^2 -\\
&\quad - \frac{\epsilon N}{2k}\phi^{-4}\sum_{mn} e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}+\gamma_{\bar{a}m}+\gamma_{\bar{a}n})r_{\bar{a}}}\,{}^3G_{o(a)(b)(c)(d)}\delta_{(a)m}\delta_{(b)r}\delta_{(c)n}\;{}^3\hat{\tilde{\pi}}^u_{(d)}\frac{d\tau(s)}{ds}\frac{d\sigma^r(s)}{ds} +\\
&\quad + {}^3\hat{\Gamma}^u_{rs}\frac{d\sigma^r(s)}{ds}\frac{d\sigma^s(s)}{ds} = 0,\\
&a^\tau = -\frac{\epsilon}{N^2}\Big({}^4\hat{R}_{\tau m\tau n}\frac{d\sigma^m}{ds}\frac{d\sigma^n}{ds}\triangle x^\tau -\\
&\quad - \Big[{}^4\hat{R}_{\tau m\tau s}\frac{d\tau}{ds} - {}^4\hat{R}_{\tau msn}\frac{d\sigma^n}{ds}\Big]\frac{d\sigma^m}{ds}\triangle x^s\Big),\\
&a^u = -\epsilon\,{}^3\hat{e}^u_{(a)}\;{}^3\hat{e}^r_{(a)}\Big(\Big[{}^4\hat{R}_{\tau r\tau n}\frac{d\tau}{ds} - {}^4\hat{R}_{rm\tau n}\frac{d\sigma^m}{ds}\Big]\frac{d\sigma^n}{ds}\triangle x^\tau -\\
&\quad - \Big[{}^4\hat{R}_{\tau r\tau s}(\frac{d\tau}{ds})^2 - ({}^4\hat{R}_{\tau rsm} + {}^4\hat{R}_{rm\tau s})\frac{d\tau}{ds}\frac{d\sigma^m}{ds} + {}^4\hat{R}_{rmsn}\frac{d\sigma^m}{ds}\frac{d\sigma^n}{ds}\Big]\triangle x^s\Big). \qquad \text{(E4)}
\end{aligned}
$$

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